

Math2058 Mathematical Analysis (I)

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1 Order structure of \mathbb{R}

Throughout this section, a number L means that it is a real number and let S be a non-empty subset of \mathbb{R} .

Definition 1.1 *Using the notation as above:*

- (i) S is said to be bounded above (resp. bounded below) if there is a number L (resp. ℓ) such that $x \leq L$ (resp. $\ell \leq x$) for all $x \in S$. In this case, such number L (resp. ℓ) is called an upper bound (resp. lower bound) for S .
Furthermore, S is said to be bounded if it is both are bounded above and bounded below.
- (ii) S is said to have a maximal element (resp. minimal element) if there is an element $M \in S$ (resp. $m \in S$) such that $x \leq M$ (resp. $m \leq x$) for all $x \in S$. In this case, write $\max S$ and $\min S$ for the maximal element and the minimal element of S respectively.

Remark 1.2 (i) It is noted that the maximum of a set may not exist even it is bounded above. For example, if let $S = \{1 - \frac{1}{n} : n = 1, 2, \dots\}$, then S is bounded above but $\max S$ does not exist.

- (ii) It is clear that $\max S$ exists if and only if $\min(-S)$ exists, where $-S = \{-x : x \in S\}$. In this case, we have $-\max S = \min(-S)$.

The following notion plays an important role in mathematics.

Definition 1.3 *Using the notation as above, a number $L \in \mathbb{R}$ (rep. ℓ) is called the **supremum** (resp. the **infimum**) of S if L is the least upper bound (resp. the greatest lower bound) for S . In this case, we write*

$$L := \sup S \quad ; \quad \ell := \inf S.$$

The following result is easy shown by the fact that a number L is an upper bound for S if and only if $-L$ is a lower bound for the set $-S$.

¹This is a note of the course Honours Mathematical Analysis I in 2022-23, 1st term

Proposition 1.4 *Using the notation as above, then $\sup S$ exists if and only if $\inf(-S)$ exists. In this case, we have*

$$-\sup S = \inf(-S).$$

The following is a very useful result for checking a number whether it is the supremum of a given set. In addition, the technique of the proof is standard.

Theorem 1.5 *Assume that $\sup S$ exists. A number $L = \sup S$ if and only if it satisfies the following two conditions.*

(i) *L is an upper bound for S .*

(ii) *For any $\varepsilon > 0$, there is an element $x_0 \in S$ such that $L - \varepsilon < x_0$.*

Similarly, if $\inf S$ exists, then a number $\ell = \inf S$ if and only if the following two conditions hold:

(i') *ℓ is a lower bound for S ;*

(ii') *For any $\varepsilon > 0$, there is an element $y_0 \in S$ such that $y_0 < \ell + \varepsilon$.*

Proof: We are going to show the case of supremum first. For showing (\Rightarrow) , assume that $L = \sup S$. It is noted that the condition (i) automatically holds by the definition of supremum. It remains to show that the condition (ii) holds. Let $\varepsilon > 0$. Then $L - \varepsilon < L$. Since L is the least upper bound for S , $L - \varepsilon$ is not an upper bound for S . Therefore, there is an element $x_0 \in S$ such that $L - \varepsilon < x_0$ as desired.

Now for showing the converse statement, assume that the conditions (i) and (ii) hold for the number L . Then by the definition of the supremum, it needs to show that if L_1 is an upper bound for S , then $L \leq L_1$. Suppose not, that is, we assume that there is an upper bound L_1 for S such that $L_1 < L$. Then $\varepsilon := 1/2(L - L_1) > 0$. The condition (ii) gives an element $x_0 \in S$ such that

$$L_1 < \frac{1}{2}(L_1 + L) = L - \varepsilon < x_0 \leq L.$$

The last statement can be obtained by considering $-S$ in the first assertion above. □

Axiom of Completeness of \mathbb{R} : *Every bounded above non-empty subset of \mathbb{R} must have the least upper bound, that is, the supremum of a bounded above non-empty subset of \mathbb{R} must exist.*

Proposition 1.6 *Let A and B be non-empty bounded above subsets of \mathbb{R} . Put $A + B := \{x + y : x \in A, y \in B\}$. Then we have $\sup(A + B) = \sup A + \sup B$.*

Proof: Note that $L_1 := \sup A$ and $L_2 := \sup B$ exist by the Axiom of Completeness. It is clear that $L_1 + L_2$ is an upper bound for the set $A + B$. By using Theorem 1.5, it suffices to show the condition (ii) in Theorem 1.5 holds. Let $\varepsilon > 0$. Then by Theorem 1.5, there are elements $a \in A$ and $b \in B$ such that $L_1 - \frac{1}{2}\varepsilon < a$ and $L_2 - \frac{1}{2}\varepsilon < b$. Hence, we have $L_1 + L_2 - \varepsilon < a + b$. Thus the condition (ii) holds for the set $A + B$. The proof is finished. □

Proposition 1.7 *If S is a bounded below non-empty subset of \mathbb{R} , then $\inf S$ must exist.*

Proof: Note that the set $-S$ is bounded above. Then by the completeness of \mathbb{R} , $\sup(-S)$ exists and hence, $\inf S = -\sup(-S)$ must exist. \square

Theorem 1.8 Archimedean Property: *For each $x \in \mathbb{R}$, there is a positive integer n such that $x < n$.*

Proof: The proof is shown by the contradiction. Suppose that there is a real number M such that $n \leq M$ for all $n \in \mathbb{Z}_+$. Thus, the set of all positive integers \mathbb{Z}_+ is bounded above. The Axiom of Completeness tells us that the supremum $L := \sup \mathbb{Z}_+$ must exist. Then by considering $\varepsilon = 1$ in Theorem 1.5, there is an element $m \in \mathbb{Z}_+$ such that $L - 1 < m$ and hence, $L < m + 1$. This implies that $n < m + 1$ for all $n \in \mathbb{Z}_+$. It leads to a contradiction because $m + 1 \in \mathbb{Z}_+$. \square

Corollary 1.9 $\inf\{1/n : n = 1, 2, \dots\} = 0$.

Proof: Let $S := \{1/n : n = 1, 2, \dots\}$. It is noted that 0 is a lower bound for the set S . By using Theorem 1.5, it needs to show that for any $\varepsilon > 0$, there is an element $a \in S$ such that $a < 0 + \varepsilon$. Now let $\varepsilon > 0$. Then by Archimedean property, there is a positive integer N such that $1/\varepsilon < N$. Thus, we have $1/N \in S$ and $1/N < \varepsilon$ as required. The proof is finished. \square

Definition 1.10 *We say that a subset A of \mathbb{R} is dense in \mathbb{R} if $(a, b) \cap A \neq \emptyset$.*

Example 1.11 *The set of all integers \mathbb{Z} is not dense in \mathbb{R} .*

The following shows that the set of rational numbers a dense subset of \mathbb{R} which \mathbb{Q} is an important dense subset.

Proposition 1.12 *For each pair of real numbers a and b with $a < b$, then we have $(a, b) \cap \mathbb{Q} \neq \emptyset$. In this case, the set of all rational numbers is dense in \mathbb{R} .*

Proof: Note that we may assume $0 < a < b$. (Why?) By using Corollary 1.9, there is a positive integer N such that $1/N < b - a$ and hence, we have $1 < Nb - Na$. On the other hand, let $p := \max\{k \in \mathbb{N} : k \leq Na\}$. This implies that $Na < p + 1$. In addition, since $Nb - Na > 1$, we have $p + 1 < Nb$. Therefore, we have $Na < p + 1 < Nb$. Thus, $\frac{p+1}{N} \in \mathbb{Q} \cap (a, b)$. The proof is finished. \square

Before showing the following proposition, we have a simple but useful observation first.

Lemma 1.13 *Let $e, f \in \mathbb{R}$. Then we have $e \leq f$ if and only if for all $\varepsilon > 0$, we have $e < f + \varepsilon$.*

Proposition 1.14 *There is a unique real number x such that $x^2 = 2$. Consequently, such real number is irrational.*

Proof: Let $S := \{x > 0 : x^2 \leq 2\}$. Note that $1 \in S$, hence, $S \neq \emptyset$. On the other hand, if $x > 2$, then $x^2 > 4$. This implies that the set S is bounded by 2 and thus, the set S is bounded above. Then the Axiom of Completeness assures that $a := \sup S$ exists. We are going to show that $a^2 = 2$ as required.

We first note that by the characterization of the sup, for each positive integer n , there is an element $x_n \in S$ such that $a - \frac{1}{n} < x_n$. This implies that

$$a^2 < (x_n + \frac{1}{n})^2 = x_n^2 + \frac{2}{n}x_n + \frac{1}{n^2} < 2 + \frac{4}{n} + \frac{1}{n^2} \quad (1.1)$$

It is noted that we have $\frac{4}{n} + \frac{1}{n^2} < \frac{5}{n}$ for all positive integer n . Therefore, we have $\inf\{\frac{4}{n} + \frac{1}{n^2} : n = 1, 2, \dots\} = 0$ because $\inf\{1/n : n = 1, 2, \dots\} = 0$. This implies that for any $\varepsilon > 0$, there is a positive integer m such that $\frac{4}{m} + \frac{1}{m^2} < \varepsilon$. Therefore, we have

$$a^2 < 2 + \varepsilon$$

for all $\varepsilon > 0$. Lemma 1.13 implies that $a^2 \leq 2$.

Finally, it remains to show that $a^2 < 2$ is impossible. Assume that $a^2 < 2$. Then by using the fact $\inf\{\frac{4}{n} + \frac{1}{n^2} : n = 1, 2, \dots\} = 0$ again, one can choose a positive integer N such that $\frac{4}{N} + \frac{1}{N^2} < 2 - a^2$, and hence, we have $a^2 + \frac{4}{N} + \frac{1}{N^2} < 2$. This implies that

$$(a + 1/N)^2 = a^2 + \frac{2}{N}a + \frac{1}{N^2} \leq a^2 + \frac{4}{N} + \frac{1}{N^2} < 2.$$

Thus, we have $(a + 1/N) \in S$ and $a < a + 1/N$. It leads to a contradiction. Therefore, $a^2 = 2$.

The uniqueness clearly follows from the fact that if $a^2 = b^2 = 2$, then we have $a^2 - b^2 = (a - b)(a + b) = 0$.

Now write $\sqrt{2} := \sup S$. Then by above we have $(\sqrt{2})^2 = 2$. Suppose that $\sqrt{2} = p/q$ is rational, for some positive integers p and q . We have $p^2 = 2q^2$. Then by the Unique Prime Factorization theorem, there are natural numbers n and s such that $p = 2^n s$ and s is not divided by 2. Similarly, there are natural numbers m and t such that $q = 2^m t$ and t is not divided by 2. Thus, we have

$$2^{2n} s^2 = p^2 = 2q^2 = 2 \cdot 2^{2m} t^2 = 2^{2m+1} t^2.$$

From this we have $2n = 2m + 1$. It is impossible. The proof is complete. \square

Theorem 1.15 For any open interval (a, b) , we have $(a, b) \cap \mathbb{Q}^c \neq \emptyset$, i.e., the set of all irrational numbers is dense in \mathbb{R} .

Proof: We may assume that $a > 0$ and hence, we have $\sqrt{2}a < \sqrt{2}b$. Since \mathbb{Q} is dense in \mathbb{R} , there is an element $r \in \mathbb{Q} \cap (\sqrt{2}a, \sqrt{2}b)$. Hence, we have

$$a < \frac{r}{\sqrt{2}} < b.$$

Since $\sqrt{2}$ is irrational and r is rational, we see that the number $\frac{r}{\sqrt{2}}$ is irrational as required. \square

2 Sequences

A *sequence* of real numbers means that it is a real-valued function x defined on \mathbb{Z}_+ (or \mathbb{N}). Write $x_n := x(n)$ for $n = 1, 2, \dots$ and $x = (x_n)$.

The following definition plays a very important role in mathematics.

Definition 2.1 We say that a sequence (x_n) is convergent if there is a number $L \in \mathbb{R}$ which satisfies the following condition:

For any $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ (depends on the choice of ε), such that

$$|x_n - L| < \varepsilon \quad \text{whenever } n \geq N.$$

In this case, we say that (x_n) converges to L and L is a **limit** of (x_n) . If such L does not exist, we say that (x_n) is divergent.

Remark 2.2 Using the notation above, we have:

- (i) A number ℓ is **Not** a limit of (x_n) if there is $\varepsilon > 0$ such that for any positive integer N , we can find a positive integer n with $n \geq N$ so that $|x_n - \ell| \geq \varepsilon$.

Warning: in this case, it does not imply that (x_n) is divergent !!!!

- (ii) The Definition 2.1 is clearly equivalent to the following statement:
there is a constant $C > 0$ such that for any $\eta > 0$, there is a positive integer N satisfying $|x_n - L| < C\eta$ as $n \geq N$.

The following is one of important properties of limits.

Proposition 2.3 If (x_n) is a convergent sequence, then its limit is unique.

In this case, we write $\lim x_n$ for “**the**” limit of (x_n) .

Proof: Let L and L' be limits of (x_n) . Then for any $\varepsilon > 0$, there are positive integers N and N' such that $|x_n - L| < \varepsilon$ for any $n \geq N$ and $|x_n - L'| < \varepsilon$ for any $n \geq N'$. Now if we choose a positive m so that $m \geq N$ and $m \geq N'$, then we have

$$|L - L'| \leq |L - x_m| + |x_m - L'| < 2\varepsilon.$$

Therefore, we have $|L - L'| < 2\varepsilon$ for all $\varepsilon > 0$. This implies that $|L - L'| = 0$ and thus, $L = L'$. Otherwise, if we choose $0 < \varepsilon < \frac{1}{4}|L - L'|$, then it leads to a contradiction. \square

Example 2.4 Show that if $x_n := \frac{n+1}{n-1}$ for $n = 2, 3, \dots$, then the sequence $\lim x_n = 1$.

Proof: Note that for each positive integer n with $n \geq 2$, we have $|x_n - 1| = \frac{2}{n-1}$. Now let $\varepsilon > 0$. Therefore, we have $|x_n - 1| < \varepsilon$ if and only if $\frac{2}{\varepsilon} + 1 < n$. The Archimedean property tells us that there is a positive integer N such that $N > \frac{2}{\varepsilon} + 1$. Hence, we have $|x_n - 1| < \varepsilon$ as $n \geq N$. The proof is complete. \square

Example 2.5 Let $x_n = (-1)^n$ for $n = 1, 2, \dots$. Show that the sequence (x_n) is divergent.

Proof: Warning: It is clear that neither 1 nor -1 both is the limit of the sequence of (x_n) . However, we cannot conclude from the Definition 2.1 that the sequence (x_n) is divergent since the sequence (x_n) may converge to the number which is other than 1 and -1 .

Now suppose that the sequence (x_n) is convergent with $L := \lim x_n$. Now if for each positive integer N , put $A_N := \{x_n : n \geq N\}$, then $A_N = \{1, -1\}$. Therefore, for any positive integer N the intersection $(L - 1/4, L + 1/4) \cap A_N$ contains at most one point. This implies that for any positive integer N , there is $m \geq N$ such that $x_m \notin (L - 1/4, L + 1/4)$, that is, $|x_m - L| \geq 1/4$. It leads to a contradiction since L is the limit of (x_n) by the assumption. \square

Example 2.6 Show that if $x_n = n$ for all $n = 1, 2, \dots$, then the sequence (x_n) is divergent.

Proof: suppose not, we assume that the sequence (x_n) converges to some number L . Then by Definition 2.1, if we consider $\varepsilon = 1$, then there is a positive integer N such that $|x_n - L| < 1$ for all $n \geq N$ and thus, $n < |L| + 1$ for all $n \geq N$. This implies that $n < |L| + 1$ for all positive integers n . This contradicts to the Archimedean property. \square

Using the similar idea as the proof of Example 2.6, one can obtain a more general result as follows.

Proposition 2.7 Every convergent sequence is bounded.

Proof: Let (x_n) be a convergent sequence with the limit L . If we take $\varepsilon = 1$ in the Definition 2.1, there is a positive integer N such that $|x_n - L| < 1$ for all $n \geq N$. Hence, we have $|x_n| < |L| + 1$ for all $n \geq N$. Thus, if we take $M := \max\{|x_1|, \dots, |x_{N-1}|, |L| + 1\}$, then we have $|x_n| \leq M$ for all $n = 1, 2, \dots$. Thus, (x_n) is bounded. \square

Proposition 2.8 Let (x_n) and (y_n) be the convergent sequences. Let $a := \lim x_n$ and $b := \lim y_n$. We have the following assertions.

- (i) $(x_n + y_n)$ is convergent with $\lim(x_n + y_n) = a + b$.
- (ii) The product $(x_n y_n)$ is convergent with $\lim x_n y_n = ab$.
- (iii) If $y_n \neq 0$ for all n and $b \neq 0$, then the sequence (x_n/y_n) is convergent and $\lim x_n/y_n = a/b$.

Proof: For showing (i): let $\varepsilon > 0$. Then there is a positive integer N such that $|x_n - a| < \varepsilon$ and $|y_n - b| < \varepsilon$ for all $n \geq N$. This implies that

$$|(x_n + y_n) - (a + b)| \leq |x_n - a| + |y_n - b| < 2\varepsilon$$

for all $n \geq N$. Thus, $(x_n + y_n)$ is convergent with $\lim(x_n + y_n) = a + b$.

For (ii), let $\varepsilon > 0$ and let N be chosen as in Part (i). Since (y_n) is convergent, (y_n) is bounded and hence, there is $M > 0$ such that $|y_n| \leq M$ for all n . Hence, the triangle inequality implies that

$$|x_n y_n - ab| \leq |x_n y_n - a y_n| + |a y_n - ab| \leq |x_n - a| |y_n| + |a| |y_n - b| \leq (M + |a|) \varepsilon$$

for all $n \geq N$. This implies that $(x_n y_n)$ is convergent and $\lim x_n y_n = ab$.

For showing (iii), it suffices to show that the sequence $(\frac{1}{y_n})$ converges to $1/b$ by using Part (ii).

Let $\varepsilon > 0$ and N be as in Part (i) again. It is noted that since $b \neq 0$, by using the Definition 2.1 there is a positive integer $N_1 > N$ such that $|y_n - b| < \frac{|b|}{2}$ for all $n \geq N_1$. This gives $|y_n| > \frac{|b|}{2}$ for all $n \geq N_1$. Hence, we have

$$\left| \frac{1}{y_n} - \frac{1}{b} \right| = \frac{|y_n - b|}{|y_n||b|} \leq \frac{2}{|b|^2} \varepsilon$$

for all $n \geq N_1$. The proof is complete. \square

Proposition 2.9 *Let (x_n) and (y_n) be the convergent sequences with the limits $a := \lim x_n$ and $b := \lim y_n$. If $x_n \leq y_n$ for all $n = 1, 2, \dots$, then $a \leq b$.*

Proof: It suffices to show that $a < b + \varepsilon$ for all $\varepsilon > 0$; otherwise, if $b < a$, then by taking $\varepsilon = a - b > 0$ we have $a < b + (a - b) = a$ which is impossible. Now let $\varepsilon > 0$. Then there is a positive integer N such that $|x_N - a| < \varepsilon$ and $|y_N - b| < \varepsilon$. This implies that

$$a - \varepsilon < x_N \leq y_N < b + \varepsilon.$$

Thus, we have $a < b + 2\varepsilon$. The proof is complete. \square

Proposition 2.10 *Let $(x_n), (y_n)$ and (z_n) be the sequences which satisfy $x_n \leq y_n \leq z_n$ for all n . If $a := \lim x_n = \lim z_n$, then (y_n) is convergent and $\lim y_n = a$.*

Proof: Let $\varepsilon > 0$. Then by the Definition 2.1, there is a positive integer N such that $|x_n - a| < \varepsilon$ and $|z_n - a| < \varepsilon$ for all $n \geq N$. This implies that

$$a - \varepsilon < x_n \leq y_n \leq z_n < a + \varepsilon$$

for all $n \geq N$. Hence, we have $|y_n - a| < \varepsilon$ for all $n \geq N$. The proof is finished. \square

Proposition 2.11 *let S be a non-empty bounded above subset of \mathbb{R} . Then a number $L = \sup S$ if and only if L is an upper bound for S and there is a sequence (x_n) in S such that $\lim x_n = L$.*

Proof: For showing (\Rightarrow) , assume $L = \sup S$. Then L is an upper bound for S by the definition. It suffices to show that there is a sequence (x_n) in S such that $\lim x_n = L$. Recall the characterization of supremum that for any $\varepsilon > 0$, there is an element $x \in S$ such that $L - \varepsilon < x$. From this for each positive integer n , there is an element $x_n \in S$ such that $L - \frac{1}{n} < x_n \leq L$. This implies that $|x_n - L| < \frac{1}{n}$ for all n and thus, $\lim x_n = L$ as required. The converse is clear due to the characterization of supremum again. \square

Definition 2.12 *A sequence (x_n) is said to be increasing (resp. decreasing) if $x_n \leq x_{n+1}$ (resp. $x_n \geq x_{n+1}$) for all n .*

Theorem 2.13 Let (x_n) be an increasing (resp. decreasing) sequence. Then (x_n) is convergent if and only if (x_n) is bounded. In this case, we have $\lim x_n = \sup\{x_n : n = 1, 2, \dots\}$ (resp. $\lim x_n = \inf\{x_n : n = 1, 2, \dots\}$).

Proof: Assume that (x_n) is increasing. It is noted that this part (\Rightarrow) is always true even (x_n) is not increasing.

Now for showing the part (\Leftarrow), assume that (x_n) is bounded. Then the set $S := \{x_n : n = 1, 2, \dots\}$ is bounded. The Axiom of Completeness tells us that $L := \sup(S)$ exists. We are going to show that $\lim x_n = L$. In fact, for any $\varepsilon > 0$, there is an element $x_N \in S$ such that $L - \varepsilon < x_N$ because $L = \sup(S)$. Since (x_n) is increasing, we have $L - \varepsilon < x_N \leq x_n \leq L$ for all $n \geq N$. Hence, $|x_n - L| < \varepsilon$ for all $n \geq N$. Therefore, (x_n) converges to L as desired.

When (x_n) is decreasing, the assertion can be obtained by considering the sequence $(-x_n)$.

□

Example 2.14 Then the following limit exists

$$e := \lim\left(1 + \frac{1}{n}\right)^n.$$

Proof: For each positive integer, let

$$x_n = \left(1 + \frac{1}{n}\right)^n.$$

They by using Proposition 2.13, it suffices to show that (x_n) is a bounded increasing sequence.

We first claim that (x_n) is increasing. In fact, by the Binomial Theorem, we see that

$$x_n = 1 + 1 + \sum_{k=2}^n \frac{n(n-1)\cdots(n-k+1)}{k!} \frac{1}{n^k} = \sum_{k=1}^n \frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right). \quad (2.1)$$

It is noted that each term in above is positive and the coefficients of $\frac{1}{k!}$ for $2 \leq k \leq n$ in x_n and x_{n+1} are

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) \quad \text{and} \quad \left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{k-1}{n+1}\right)$$

respectively. From this we see that $x_n \leq x_{n+1}$ for all n and thus, the sequence (x_n) is increasing.

It remains to show that (x_n) is bounded. In fact, for each $2 \leq k \leq n$ we have

$$\frac{1}{k!} \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-1}{n}\right) < \frac{1}{2^k}.$$

Then

$$x_n < 1 + 1 + \sum_{k=1}^n \frac{1}{2^k} < 3.$$

The proof is complete. □

Remark 2.15 The limit e in the Example 2.14 above is very important in mathematics which is called the natural base today. It was first induced by Euler. In fact, $x_n := \left(1 + \frac{1}{n}\right)^n$ is motivated by the Compound interest formula.

Theorem 2.16 Nested Intervals Theorem Let $(I_n := [a_n, b_n])$ be a sequence of closed and bounded intervals. Assume that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$. Then we have $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. Furthermore, if we further assume that $\lim_n(b_n - a_n) = 0$, then there is a unique real number c such that $\bigcap_{n=1}^{\infty} I_n = \{c\}$.

Proof: It is noted that since (I_n) is a decreasing sequence of closed and bounded intervals, we have

$$a_1 \leq a_2 \leq \dots \leq a_n < b_n \leq b_{n-1} \leq \dots \leq b_2 \leq b_1$$

for all positive integers n . Therefore, (a_n) and (b_n) are bounded and they are increasing and decreasing and respectively. This implies that (x_n) and (y_n) both are convergent and $a := \lim a_n = \sup\{a_n : n = 1, 2, \dots\}$ and $b := \lim b_n = \inf\{b_n : n = 1, 2, \dots\}$. In addition, we have $a \leq b$ because $a_n \leq b_n$ for all n . Thus, if we fix some c such that $a \leq c \leq b$, then $c \in \bigcap_{n=1}^{\infty} I_n$ as desired because we have $a_n \leq a \leq c \leq b \leq b_n$ for all n .

It remains to show $\bigcap_{n=1}^{\infty} I_n = \{c\}$ if $\lim(b_n - a_n) = 0$. In fact, if $c' \in \bigcap_{n=1}^{\infty} I_n$, then we have $|c - c'| \leq |b_n - a_n|$ for all n . This implies that $|c - c'| = 0$ and thus, $c = c'$. The proof is finished. \square

Remark 2.17 The assumption of the boundedness and closeness of the intervals I_n cannot be removed in the Nest Intervals Theorem.

For example, if $I_n := (0, \frac{1}{n})$ and $J_n := [n, \infty)$, for all $n = 1, 2, \dots$, then $\bigcap I_n = \bigcap J_n = \emptyset$.

3 Subsequences

Definition 3.1 A subsequence $(x_{n_k})_{k=1}^{\infty}$ of a sequence (x_n) means that $(n_k)_{k=1}^{\infty}$ is a sequence of positive integers satisfying $n_1 < n_2 < \dots < n_k < n_{k+1} < \dots$, that is, such sequence (n_k) can be viewed as a strictly increasing function $\mathbf{n} : k \in \{1, 2, \dots\} \mapsto n_k \in \{1, 2, \dots\}$.

Remark 3.2 In this case, note that for each positive integer N , there is $K \in \mathbb{N}$ such that $n_K \geq N$ and thus we have $n_k \geq N$ for all $k \geq K$.

Proposition 3.3 If (x_n) is a convergent sequence, then any subsequence (x_{n_k}) of (x_n) converges to the same limit. In this case, we have $\lim_k x_{n_k} = \lim x_n$.

Proof: We assume that $\lim x_n = a \in \mathbb{R}$ exists. Let (x_{n_k}) be a subsequence of (x_n) . We claim that $\lim x_{n_k} = a$. Let $\varepsilon > 0$. In fact, since $\lim x_n = a$, there is a positive integer N such that $|a - x_n| < \varepsilon$ for all $n \geq N$. Note that by the definition of a subsequence, there is a positive integer K such that $n_k \geq N$ for all $k \geq K$. Hence, we see that $|a - x_{n_k}| < \varepsilon$ for all $k \geq K$. Thus we have $\lim_{k \rightarrow \infty} x_{n_k} = a$. The proof is complete. \square

Theorem 3.4 Bolzano-Weierstrass Theorem (write B-W Theorem for short):
Every bounded sequence has a convergent subsequence.

Proof: We give two different proofs in here, however, each proof basically is due to the Axiom of Completeness.

Let (x_n) be a bounded sequence and put $X := \{x_n : n = 1, 2, \dots\}$. The Theorem clearly holds if X is a finite set. In fact in this case, there must have an element x_m appears infinite many times. Hence, we can choose a subsequence (x_{n_k}) so that $x_{n_k} \equiv x_m$ for all $k = 1, 2, \dots$

Thus we may assume that the set X is infinite.

Method 1:

Since (x_n) is bounded, there is a closed and bounded interval $I_1 = [a_1, b_1]$ such that $x_n \in I_1$ for all n . Put $x_{n_1} := x_1$.

It is noted that one of the following sets must be infinite:

$$A_2 := \{n \in \mathbb{Z}_+ : x_n \in [a_1, \frac{a_1 + b_1}{2}]\}; \quad B_2 := \{n \in \mathbb{Z}_+ : x_n \in [\frac{a_1 + b_1}{2}, b_1]\}.$$

We may assume that the set A_2 is infinite. Hence there is an element $n_2 \in A_2$ such that $n_1 < n_2$. Put $I_2 := [a_2, b_2] = [a_1, \frac{a_1 + b_1}{2}]$. Thus $x_{n_2} \in I_2$. Similarly, one of the following sets is infinite:

$$A_3 := \{n \in \mathbb{Z}_+ : x_n \in [a_2, \frac{a_2 + b_2}{2}]\}; \quad B_3 := \{n \in \mathbb{Z}_+ : x_n \in [\frac{a_2 + b_2}{2}, b_2]\}.$$

In addition, we may assume that the set A_3 is infinite. Hence, there is an element $n_3 \in A_3$ such that $n_1 < n_2 < n_3$. Put $I_3 := [a_3, b_3] = [a_2, \frac{a_2 + b_2}{2}]$. Thus, $x_{n_3} \in I_3$. By repeating the same step, we can get a decreasing sequence of a closed and bounded intervals $I_k = [a_k, b_k]$ and a subsequence (x_{n_k}) of (x_n) such that the following conditions hold:

1. $\lim(b_k - a_k) = \lim \frac{1}{2^k}(b_1 - a_1) = 0$.
2. $x_{n_k} \in I_k$ for all $k = 1, 2, \dots$

The Nest Intervals Theorem tells us that there is a number c such that $c \in I_k$ for all k and hence, we have $|x_{n_k} - c| \leq (b_k - a_k) = \frac{1}{2^k}(b_1 - a_1) \rightarrow 0$. Therefore the subsequence (x_{n_k}) is convergent as required. The proof is finished.

Method 2

This method is the Weierstrass' original proof.

Recall our assumption that the set $X = \{x_n : n = 1, 2, \dots\}$ is infinite. Let

$$S := \{x \in \mathbb{R} : (x, \infty) \cap X \text{ is infinite}\}.$$

We first note that since (x_n) is bounded, there are real numbers m and M so that $m \leq x_n \leq M$ for all n . Since the set $X = \{x_n : n = 1, 2, \dots\}$ is infinite, the set S is a bounded above non-empty set because $m \in S$ and $x \leq M$ for all $x \in S$. The Axiom of Completeness implies that $L := \sup(S)$ must exist. We want to show that there is a subsequence (x_{n_k}) of (x_n) which converges to L .

Claim: For any $\varepsilon > 0$, there is an element $u \in S$ such that $|u - L| < \varepsilon$ and $(u, L + \varepsilon] \cap X$ is infinite.

In fact, if let $\varepsilon > 0$, then by the characterization of the supremum there is an element $u \in S$ such that $L - \varepsilon < u$. Since $u \in S$, we have $(u, \infty) \cap X$ is infinite. It implies that the set $(u, L + \varepsilon] \cap X$ must be infinite, otherwise, $(L + \varepsilon, \infty) \cap X$ is infinite and thus, $L + \varepsilon \in S$ by the construction of S . It leads to a contradiction because L is an upper bound for S . Thus, the **Claim** follows.

Now for $\varepsilon = 1$, then there is $u_1 \in S$ such that $L - 1 < u_1 < L + 1$. Then by the Claim above, choose $x_{n_1} \in (u_1, L + 1]$ and hence, $L - 1 < x_{n_1} \leq L + 1$. Next, we considering $\varepsilon = 1/2$, then there is an element $u_2 \in S$ such that the set $(u_2, L + 1/2]$ is infinite by the Claim above again. Therefore we can find x_{n_2} such that $n_1 < n_2$ and $L - 1/2 < u_2 < x_{n_2} \leq L + 1/2$. By repeating the same step and considering $\varepsilon = \frac{1}{k}$ for $k = 1, 2, \dots$ in the **Claim** above, we can get a subsequence (x_{n_k}) of (x_n) such that $L - \frac{1}{k} < x_{n_k} \leq L + \frac{1}{k}$ for all $k = 1, 2, \dots$. Therefore, (x_{n_k}) is a convergent subsequence of (x_n) with the limit L . The proof is complete. \square

Remark 3.5 The assumption of the boundedness of (x_n) cannot be removed. For example, let $x_n = n$ for all $n = 1, 2, \dots$. Then (x_n) does not have a convergent subsequence because $|x_n - x_m| \geq 1$ for $n \neq m$.

Proposition 3.6 Let (x_n) be a bounded sequence. For each positive integer n , put

$$a_n := \inf\{x_k : k \geq n\} \quad \text{and} \quad b_n := \sup\{x_k : k \geq n\}.$$

Then we have the following assertions.

- (i) The limits $\lim a_n$ and $\lim b_n$ always exist with $\lim a_n \leq \lim b_n$. In this case, we write $\underline{\lim} x_n := \lim a_n$ (called the *lim inf* of (x_n)) and $\overline{\lim} x_n = \lim b_n$ (called the *lim sup* of (x_n)).
- (ii) (x_n) is convergent if and only if $\underline{\lim} x_n = \overline{\lim} x_n$. In this case, we have $\lim x_n = \underline{\lim} x_n = \overline{\lim} x_n$.
- (iii) There exists a subsequence (x_{n_k}) of (x_n) such that $\lim x_{n_k} = \overline{\lim} x_n$. Consequently, the Bolzano-Weierstrass Theorem holds.

Proof: For showing part (i), we note that if $a_n \leq x_k$ for all $k \geq n$, then $a_n \leq x_k$ for $k \geq n+1$. Thus, we have $a_n \leq a_{n+1}$ for all n . Similarly, we have $b_{n+1} \geq b_n$. Thus, we have $a_1 \leq \dots \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq \dots \leq b_1$ for all n . This implies that (a_n) and (b_n) both are bounded monotone sequences. Therefore, $\lim a_n$ and $\lim b_n$ both exist. In fact, we have

$$\underline{\lim} x_n = \sup_n \inf_{k \geq n} x_k \leq \overline{\lim} x_n = \inf_n \sup_{k \geq n} x_k.$$

For part (ii), we first assume that $l := \lim x_n$ exists. Thus, for any $\varepsilon > 0$, there is a positive integer N such that $l - \varepsilon < x_n < l + \varepsilon$ for all $n \geq N$. Then by the definition of a_n and b_n , we have

$$l - \varepsilon \leq a_n \leq b_n \leq l + \varepsilon$$

for all $n \geq N$. Thus, we have $|b_n - a_n| \leq 2\varepsilon$ for all $n \geq N$. By taking $n \rightarrow \infty$, this gives $|\overline{\lim} x_n - \underline{\lim} x_n| \leq 2\varepsilon$ for all $\varepsilon > 0$, and hence, we have $\overline{\lim} x_n = \underline{\lim} x_n$.

Now for showing the converse (\Leftarrow), we assume that we have $l := \overline{\lim} x_n = \underline{\lim} x_n$. Then for any ε , there is a positive integer N so that $l - \varepsilon < a_n \leq b_n < l + \varepsilon$ for all $n \geq N$. Since we always have $a_n \leq x_k \leq b_n$ for all $k \geq n$. Therefore, we have $l - \varepsilon < x_k < l + \varepsilon$ for all $k \geq N$ and hence, $\lim x_k = l$.

For proving part (iii), we are going to construct a subsequence (x_{n_k}) of (x_n) so that $\lim x_{n_k} = \overline{\lim} x_n$. Let $L := \overline{\lim} x_n$. It is noted that for any $\varepsilon > 0$, there is a positive integer N so that $L - \varepsilon < b_n := \sup_{k \geq n} x_k < L + \varepsilon$ for all $n \geq N$. This implies that $x_k < L + \varepsilon$ for all $k \geq N$.

If we fix $n \geq N$, since $L - \varepsilon < b_n$ for all $n \geq N$, we can choose $\eta > 0$ such that $L - \varepsilon < b_n - \eta$. Using the characterization of surpemum, we have $L - \varepsilon < b_n - \eta < x_m$ for some $m \geq n$. Therefore, we have shown that

$$\forall \varepsilon > 0, \exists N, \forall n \geq N, \exists m \geq n \quad \text{so that} \quad L - \varepsilon < x_m < L + \varepsilon. \quad (3.1)$$

Now for considering $\varepsilon = 1$ in 3.1, there is N_1 so that $L - 1 < x_{n_1} < L + 1$ for some $n_1 \geq N_1$. Next, for considering $\varepsilon = 1/2$ in 3.1, there is N_2 so that for any $n \geq N_2$, we have $L - 1/2 < x_m < L + 1/2$ for some $m \geq n$. Thus, if we choose $n > N_2$ and $n > n_1$, then there is $n_2 \geq n$ so that $n_2 > n_1$ and $L - 1/2 < x_{n_2} < L + 1/2$.

Similarly, if we take $\varepsilon = 1/3$, there is a positive integer N_3 so that for any $n \geq N_3$ we have $L - 1/3 < x_m < L + 1/3$ for some $m \geq n$. Therefore, if we take $n > N_3$ and $n > n_2$, then there is $n_3 \geq n$ such that $L - 1/3 < x_{n_3} < L + 1/3$ and $n_3 > n_2$.

To repeat the same steps, we get a strictly increasing sequence of positive integers (n_k) so that $L - 1/k < x_{n_k} < L + 1/k$ for all k . Thus, (x_{n_k}) is a convergent subsequence with the limit L . The proof is complete. \square

Proposition 3.7 *Let (x_n) and (y_n) be bounded sequences. Then we have*

$$(i) \overline{\lim}(-a_n) = -\underline{\lim}a_n.$$

$$(ii) \overline{\lim}(ax_n) = a\overline{\lim}x_n \text{ for } a \geq 0.$$

$$(iii) \underline{\lim}x_n + \underline{\lim}y_n \leq \underline{\lim}(x_n + y_n) \leq \overline{\lim}(x_n + y_n) \leq \overline{\lim}x_n + \overline{\lim}y_n$$

Proof: Parts (i) and (ii) are clear. We want to show part (iii) and claim that

$$\overline{\lim}(x_n + y_n) \leq \overline{\lim}x_n + \overline{\lim}y_n.$$

Let $b := \overline{\lim}x_n$ and $c := \overline{\lim}y_n$. Let $\varepsilon > 0$. Then there is a positive integer N such that $b_n < b + \varepsilon$ and $c_n < c + \varepsilon$ for all $n \geq N$. This implies that $x_k + y_k \leq b_n + c_n < b + c + 2\varepsilon$ for all $k \geq n \geq N$. Therefore, we have $\sup_{k \geq n}(x_k + y_k) < b + c + 2\varepsilon$ for all $n \geq N$ and thus, $\overline{\lim}(x_n + y_n) = \lim_n \sup_{k \geq n}(x_k + y_k) < b + c + 2\varepsilon$ for all $\varepsilon > 0$. This gives $\overline{\lim}(x_n + y_n) = \lim_n \sup_{k \geq n}(x_k + y_k) < b + c$ as desired.

By considering the sequences $(-x_n)$ and $(-y_n)$ in above, we see that $\underline{\lim}x_n + \underline{\lim}y_n \leq \underline{\lim}(x_n + y_n)$. the proof is complete. \square

Remark 3.8 It is noted that in general we don't have the equality $\overline{\lim}(x_n + y_n) = \overline{\lim}x_n + \overline{\lim}y_n$. For example, if we let $x_n = (-1)^{n+1}$ and $y_n = (-1)^n$, then $\overline{\lim}(x_n + y_n) < \overline{\lim}x_n + \overline{\lim}y_n$.

4 Compact Sets

Motivated by the Bolzano-Weierstrass Theorem, the following notation plays a very important role in Mathematics.

Definition 4.1 *A subset A of \mathbb{R} is said to be compact if for any sequence (x_n) in A , there is a convergent subsequence (x_{n_k}) of (x_n) such that $\lim x_{n_k} \in A$.*

Example 4.2 *Clearly, \mathbb{R} and $(0, 1)$ are not compact.*

Proposition 4.3 *Every closed and bounded interval is compact.*

Proof: Recall a closed and bounded interval that it is a set $[a, b] := \{x : a \leq x \leq b\}$ for some $-\infty < a < b < \infty$.

Let (x_n) be a sequence in $[a, b]$. Then (x_n) is a bounded sequence. The Bolzano-Weierstrass Theorem gives a convergent subsequence (x_{n_k}) . It is noted since $a \leq x_{n_k} \leq b$ for all $k = 1, 2, \dots$, we have $a \leq \lim x_{n_k} \leq b$. Thus, $\lim x_{n_k} \in [a, b]$ as desired. \square

Remark 4.4 However, a compact set need not be a closed and bounded interval. For example, $[0, 1] \cup \{2\}$ is a compact set but it is not an interval.

In the remainder of this section, we give a characterization of a compact set.

Definition 4.5 Let A be a subset of \mathbb{R} . A point x is called a *limit point* (or *cluster point*) of A if for any $\varepsilon > 0$, there is an element $a \in A$ such that $0 < |x - a| < \varepsilon$, i.e., there is an element $a \in A$ with $x \neq a$ such that $|x - a| < \varepsilon$. We write $D(A)$ for the set of all limit points of A .

Furthermore, A is said to be *closed* if $D(A) \subseteq A$.

Example 4.6

- (i) If $A = (0, 1] \cup \{2\}$, then $D(A) = [0, 1]$. Hence, A is not closed since $0 \in D(A) \setminus A$.
- (ii) If $A = \mathbb{Z}$, then $D(A) = \emptyset$ and thus, \mathbb{Z} is a closed set.

Proposition 4.7 Let A be a subset of \mathbb{R} . Then the following statements are equivalent.

- (i) A is closed.
- (ii) If (x_n) is a sequence in A and is convergent, then $\lim x_n \in A$.

Proof: For (i) \Rightarrow (ii), assume that A is closed but the condition (ii) does not hold. Then there is a convergent sequence (x_n) in A but the limit $l := \lim x_n \notin A$. Since A is closed, $D(A) \subseteq A$. Thus, l is not a limit point of A . This implies that there is $\delta > 0$ so that $((l - \delta, l + \delta) \setminus \{l\}) \cap A = \emptyset$. Since $\lim x_n = l$, there is a positive integer N such that $|x_N - l| < \delta$. Note that we have $l \neq x_N$ because $l \notin A$. Hence, $x_N \in ((l - \delta, l + \delta) \setminus \{l\}) \cap A$ which leads to a contradiction. Therefore, (ii) holds.

For (ii) \Rightarrow (i), let $z \in D(A)$. Then for any $\varepsilon > 0$, there is an element $x \in A$ such that $0 < |x - z| < \varepsilon$. Therefore, for each positive integer n , there is an element $x_n \in A$ such that $0 < |x_n - z| < 1/n$ and thus, $z := \lim x_n$. The assumption (ii) implies that $z \in A$. Therefore, $D(A) \subseteq A$. The proof is complete. \square

Definition 4.8 For a subset A of \mathbb{R} , put

$$\bar{A} = A \cup D(A).$$

The set \bar{A} is called the *closure* of A .

Example 4.9 We have the following examples.

1. $\overline{(0, 1]} = [0, 1]$.
2. $\overline{\mathbb{Q}} = \mathbb{R}$.
3. $\overline{\mathbb{Z}} = \mathbb{Z}$.

Proposition 4.10 Let A be a subset of \mathbb{R} . Then we have the following assertions.

1. \bar{A} is closed.
2. A is closed if and only if $\bar{A} = A$.
3. $z \in \bar{A}$ if and only if for any $\delta > 0$, there is an element $a \in A$ so that $|z - a| < \delta$ if and only if there is a convergent sequence (x_n) in A so that $z = \lim x_n$.
4. \bar{A} is the smallest closed set containing A , i.e., if B is a closed set containing A , then $\bar{A} \subseteq B$.

Proof: For showing part (1), we need to show that $D(\bar{A}) \subseteq \bar{A}$. Suppose not, assume that there is an element $z \in D(\bar{A})$ but $z \notin \bar{A}$. Since $z \notin \bar{A}$, there is $\delta > 0$ such that $(z - \delta, z + \delta) \cap A = \emptyset$. On the other hand, there is an element $b \in (z - \delta, z + \delta) \cap \bar{A}$ because $z \in D(\bar{A})$. Now choose $r > 0$ such that $(b - r, b + r) \subseteq (z - \delta, z + \delta)$. Using the definition of limit points again, we can find some element $a \in A$ such that $a \in (b - r, b + r)$ and thus, $a \in (z - \delta, z + \delta) \cap A$. It leads to a contradiction because $(z - \delta, z + \delta) \cap A = \emptyset$ by the choice of δ .

Parts (2)-(4) can be shown by the definition of limit points directly. Try to do it by yourself. \square

Recall that a subset A of \mathbb{R} is said to be dense in \mathbb{R} if for any open interval I , we have $I \cap A \neq \emptyset$.

Proposition 4.11 *Let A be a subset of \mathbb{R} . Then A is dense in \mathbb{R} if and only if $\bar{A} = \mathbb{R}$.*

Proof: For showing (\Rightarrow) : assume that A is a dense set. Let $z \in \mathbb{R}$. Then for any $\delta > 0$, we have $(z - \delta, z + \delta) \cap A \neq \emptyset$ by the definition of a dense set. Hence, there is $a \in A$ such that $|z - a| < \delta$. Thus, $z \in \bar{A}$ by Proposition 4.10(3) above.

Conversely, assume that $\bar{A} = \mathbb{R}$. Let I be an open interval. We want to show $I \cap A$ is non-empty. Fix an element $z \in I$. Since I is an open interval, we can choose $\delta > 0$ such that $(z - \delta, z + \delta) \subseteq I$. Since $\bar{A} = \mathbb{R}$, by using Proposition 4.10(3) again, there is an element $a \in A$ such that $|z - a| < \delta$. Therefore, $a \in (z - \delta, z + \delta) \cap A$ and hence, $I \cap A \neq \emptyset$. The proof is finished. \square

Theorem 4.12 *Let A be a subset of \mathbb{R} . Then A is compact if and only if A is a closed and bounded subset.*

Proof: For showing the necessary part, we assume that A is compact.

We first claim that A is bounded. Suppose that A is unbounded. If we fix an element $x_1 \in A$, then there is $x_2 \in A$ such that $|x_1 - x_2| > 1$. Using the unboundedness of A , we can find an element $x_3 \in A$ such that $|x_3 - x_k| > 1$ for $k = 1, 2$. To repeat the same step, we can find a sequence (x_n) in A such that $|x_n - x_m| > 1$ for $n \neq m$. Thus A has no convergent subsequence. Thus A must be bounded.

Finally, we show that A is closed. Let (x_n) be a sequence in A and it is convergent. It needs to show that $\lim_n x_n \in A$. Note that since A is compact, (x_n) has a convergent subsequence (x_{n_k}) such that $\lim_k x_{n_k} \in A$. Then by Proposition 3.3, we see that $\lim_n x_n = \lim_k x_{n_k} \in A$. The proof is finished.

Conversely, we suppose that A is closed and bounded. Let (x_n) be a sequence in A and thus (x_n) is a bounded sequence in \mathbb{R} . Then by the Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k}) . Since A is closed, $\lim_k x_{n_k} \in A$. Therefore, A is compact.

□

Example 4.13 Let $A = \{1/n : n = 1, 2, \dots\} \cup \{0\}$. Then A is a compact set. A is clearly bounded. Then by Theorem 4.12, it suffices to show that the set A is closed. Clearly, $0 \in D(A)$. We are going to show $D(A) = \{0\}$. In fact, if $z \neq 0$, clearly we can find some $r > 0$ such that the intersection $(z - r, z + r) \cap A$ contains at most one point. Therefore, if $z \neq 0$, then $z \notin D(A)$. Thus, $D(A) = \{0\}$. Hence, the set A is closed as desired.

In the rest of this section, we are going to use another description of a compact set.

For convenience, we call a collection of open intervals $\{J_\alpha : \alpha \in \Lambda\}$ an *open intervals cover* of a given subset A of \mathbb{R} , where Λ is an arbitrary non-empty index set, if each J_α is an open interval (not necessary bounded) and

$$A \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha.$$

Theorem 4.14 Heine-Borel Theorem: Any closed and bounded interval $[a, b]$ satisfies the following condition which is called *the Heine-Borel Property*.

(HB) Given any open intervals cover $\{J_\alpha\}_{\alpha \in \Lambda}$ of $[a, b]$, there are finitely many $J_{\alpha_1}, \dots, J_{\alpha_N}$ such that $[a, b] \subseteq J_{\alpha_1} \cup \dots \cup J_{\alpha_N}$

Proof: Suppose that $[a, b]$ does not satisfy Heine-Borel Property. Then there is an open intervals cover $\{J_\alpha\}_{\alpha \in \Lambda}$ of $[a, b]$ but it has no finite sub-cover. Let $I_1 := [a_1, b_1] = [a, b]$ and m_1 the mid-point of $[a_1, b_1]$. Then by the assumption, $[a_1, m_1]$ or $[m_1, b_1]$ cannot be covered by finitely many J_α 's. We may assume that $[a_1, m_1]$ cannot be covered by finitely many J_α 's. Put $I_2 := [a_2, b_2] = [a_1, m_1]$. To repeat the same steps, we can obtain a sequence of closed and bounded intervals $I_n = [a_n, b_n]$ with the following properties:

- (a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$;
- (b) $\lim_n (b_n - a_n) = 0$;
- (c) each I_n cannot be covered by finitely many J_α 's.

Then by the Nested Intervals Theorem, there is an element $\xi \in \bigcap_n I_n$ such that $\lim_n a_n = \lim_n b_n = \xi$. In particular, we have $a = a_1 \leq \xi \leq b_1 = b$. Hence, there is $\alpha_0 \in \Lambda$ such that $\xi \in J_{\alpha_0}$. Since J_{α_0} is open, there is $\varepsilon > 0$ such that $(\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. On the other hand, there is $N \in \mathbb{N}$ such that a_N and b_N in $(\xi - \varepsilon, \xi + \varepsilon)$ because $\lim_n a_n = \lim_n b_n = \xi$. Thus we have $I_N = [a_N, b_N] \subseteq (\xi - \varepsilon, \xi + \varepsilon) \subseteq J_{\alpha_0}$. It contradicts to the Property (c) above. The proof is complete.

□

Remark 4.15 The assumption of the closeness and boundedness of an interval in Heine-Borel Theorem is essential.

For example, notice that $\{J_n := (1/n, 1) : n = 1, 2, \dots\}$ is an open interval covers of $(0, 1)$ but you cannot find finitely many J_n 's to cover the open interval $(0, 1)$.

Lemma 4.16 A subset A is a closed subset of \mathbb{R} if and only if for each element $x \notin A$, there is $r > 0$ such that $(x - r, x + r) \cap A = \emptyset$.

The following is a very important feature of a compact set.

Theorem 4.17 Let A be a subset of \mathbb{R} . Then the following statements are equivalent.

(i) For any open intervals cover $\{J_\alpha\}_{\alpha \in \Lambda}$ of A , we can find finitely many $J_{\alpha_1}, \dots, J_{\alpha_N}$ such that $A \subseteq J_{\alpha_1} \cup \dots \cup J_{\alpha_N}$.

(ii) A is compact.

(iii) A is closed and bounded.

Proof: The result will be shown by the following path

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).$$

For $(i) \Rightarrow (ii)$, assume that the condition (i) holds but A is not compact. Then there is a sequence (x_n) in A such that (x_n) has no subsequence which has the limit in A . Put $X = \{x_n : n = 1, 2, \dots\}$. Then X is infinite. Note that for each element $a \in A$, there is $\delta_a > 0$ such that $J_a := (a - \delta_a, a + \delta_a) \cap X$ is finite. Indeed, if there is an element $a \in A$ such that $(a - \delta, a + \delta) \cap X$ is infinite for all $\delta > 0$, then (x_n) has a convergent subsequence with the limit a . On the other hand, we have $A \subseteq \bigcup_{a \in A} J_a$. Then by the compactness of A , we can find finitely many a_1, \dots, a_N such that $A \subseteq J_{a_1} \cup \dots \cup J_{a_N}$. Hence, we have $X \subseteq J_{a_1} \cup \dots \cup J_{a_N}$. Then by the choice of J_a 's, X must be finite. This leads to a contradiction. Therefore, A must be compact.

The implication $(ii) \Rightarrow (iii)$ follows immediately from Theorem 4.12.

Finally we want to show $(iii) \Rightarrow (i)$. Suppose that A is closed and bounded. Then we can find a closed and bounded interval $[a, b]$ such that $A \subseteq [a, b]$. Now let $\{J_\alpha\}_{\alpha \in \Lambda}$ be an open intervals cover of A . Note that for each element $x \in [a, b] \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$ since A is closed by using Lemma 4.16. If we put $I_x = (x - \delta_x, x + \delta_x)$ for $x \in [a, b] \setminus A$, then we have

$$[a, b] \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha \cup \bigcup_{x \in [a, b] \setminus A} I_x.$$

Using the Heine-Borel Theorem 4.14, we can find finitely many J_α 's and I_x 's, say $J_{\alpha_1}, \dots, J_{\alpha_N}$ and I_{x_1}, \dots, I_{x_K} , such that $A \subseteq [a, b] \subseteq J_{\alpha_1} \cup \dots \cup J_{\alpha_N} \cup I_{x_1} \cup \dots \cup I_{x_K}$. Note that $I_x \cap A = \emptyset$ for each $x \in [a, b] \setminus A$ by the choice of I_x . Therefore, we have $A \subseteq J_{\alpha_1} \cup \dots \cup J_{\alpha_N}$ and hence A is compact.

The proof is complete. □

Remark 4.18 In fact, the condition in Theorem 4.17(i) is the usual definition of a *compact set* for a general topological space. More precisely, if a set A satisfies the Definition 4.1, then A is said to be *sequentially compact*. Theorem 4.17 tells us that the notation of the compactness and the sequentially compactness are the same as in the case of a subset of \mathbb{R} . However, these two notations are different for a general topological space.

5 Cauchy sequences

The following notation is the landmark in the development of the 20th century mathematics.

Definition 5.1 A sequence (x_n) is called a Cauchy sequence if it satisfies the following condition:

for any $\varepsilon > 0$, there is a positive integer N so that $|x_m - x_n| < \varepsilon$ whenever $m, n \geq N$.

Remark 5.2 According to the definition of a Cauchy sequence, a sequence (x_n) is not a Cauchy sequence if there is $\varepsilon > 0$ so that for any positive integer N , we can find some $m, n \geq N$ such that $|x_m - x_n| \geq \varepsilon$.

Theorem 5.3 Cauchy Criterion: A sequence (x_n) is convergent if and only if it is a Cauchy sequence.

Proof: The necessary part is clear. In fact, if (x_n) is a convergent sequence with the limit L , then for any $\varepsilon > 0$, there is a positive integer N such that $|x_n - L| < \varepsilon$ for all $n \geq N$. Therefore, we have

$$|x_m - x_n| \leq |x_m - L| + |L - x_n| < 2\varepsilon \quad \text{as } m, n \geq N.$$

Conversely, we assume that (x_n) is a Cauchy sequence.

We first Claim that (x_n) is a bounded sequence. In fact, since (x_n) is a Cauchy, we can find a positive integer N_1 such that $|x_m - x_{N_1}| < 1$ for all $m \geq N_1$ and thus, $|x_m| < 1 + |x_{N_1}|$ for all $m \geq N_1$. Therefore, we have $|x_m| \leq \max(|x_1|, \dots, |x_{N_1-1}|, |x_{N_1}| + 1)$ for all positive integers m .

The Bolzano-Weierstrass Theorem tells us that (x_n) has a convergent subsequence (x_{n_k}) . Let $L := \lim_k x_{n_k}$. If we show that L is the limit of (x_n) , then the proof is finished.

Let $\varepsilon > 0$. Then there is a positive integer N such that $|x_m - x_n| < \varepsilon$ as $m, n \geq N$. On the other hand, since $L = \lim_k x_{n_k}$, we can choose K large enough such that $|L - x_{n_K}| < \varepsilon$ and $n_K > N$. This implies that for any $n \geq N$, we have

$$|x_n - L| < |x_n - x_{n_K}| + |x_{n_K} - L| < 2\varepsilon.$$

The proof is complete. □

Example 5.4 Let $s_n = \sum_{k=1}^n 1/k$. Then (s_n) is not a Cauchy sequence and thus, (s_n) is divergent.

In fact, it is noted that for $n \leq m$, we have

$$|s_m - s_n| = \frac{1}{n+1} + \dots + \frac{1}{m} \geq \frac{m-n}{m}.$$

Hence, we always have $|s_{2n} - s_n| \geq \frac{1}{2}$ for all n . Thus, if we take $\varepsilon = 1/2$, then for any positive integer N by taking $n = N$ and $m = 2N$, we have $|s_{2N} - s_N| > 1/2 = \varepsilon$. Hence, (s_n) is not a Cauchy sequence.

Remark 5.5 A sequence (x_n) properly converges to $+\infty$ (resp. $-\infty$) if for any $M > 0$, there is a positive integer N so that $x_n > M$ (resp. $x_n < -M$) for all $n \geq M$. In this case, we write $\lim x_n = \infty$ (resp. $\lim x_n = -\infty$). **Warning!!!** In this case, the sequence (x_n) is still divergent since ∞ is **NOT** a real number, hence, ∞ is not the limit of (x_n) .

Note that the sequence (s_n) in Example 5.4 properly converges to $+\infty$. From this we see that the sequence $(\sum_{k=1}^n \frac{1}{n^\alpha})_{n=1}^\infty$ also diverges properly to $+\infty$ if $\alpha \leq 1$.

However, a divergent sequence may not converge properly to ∞ , for example, if we take $x_n = 0$ as n is odd; otherwise, $x_n = n$.

Example 5.6 Let $t_n = \sum_{k=1}^n \frac{1}{k^2}$. Then the sequence (t_n) is convergent. Using the Cauchy Theorem, we need to show that (t_n) is a Cauchy sequence.

It is noted that for $n \leq m$, we have

$$|t_m - t_n| = \sum_{k=n+1}^m \frac{1}{k^2} \leq \sum_{k=n+1}^m \frac{1}{(k-1)k} = \sum_{k=n+1}^m (\frac{1}{k-1} - \frac{1}{k}) = \frac{1}{n} - \frac{1}{m} < \frac{1}{n}.$$

Thus, if we are given $\varepsilon > 0$, then we choose a positive integer N so that $\frac{1}{n} < \varepsilon$ for all $n \geq N$. Therefore, $|t_m - t_n| < \varepsilon$ whenever $m \geq n \geq N$. The proof is complete.

Remark 5.7 We have the following implications in \mathbb{R} .

Axiom of Completeness \Rightarrow Bounded Monotone Convergent Theorem (Theorem 2.13) \Rightarrow
Nested Intervals Theorem \Rightarrow Bolzano-Weierstrass Theorem \Rightarrow Cauchy Theorem.

Everything is due to the Axiom of Completeness.

6 Appendix: Bolzano-Weierstrass Theorem and Cauchy Criterion in \mathbb{R}^m

Throughout this section, for each element $x \in \mathbb{R}^m$, we write $x := (x(1), \dots, x(m))$ and put

$$\|x\| := \sqrt{\sum_{k=1}^m |x(k)|^2}.$$

We call $\|x\|$ the norm of x . Clearly, we have

$$\max_{1 \leq k \leq m} |x(k)| \leq \|x\| \leq \sqrt{m} \max_{1 \leq k \leq m} |x(k)| \tag{6.1}$$

for all $x \in \mathbb{R}^m$.

For each element x and y in \mathbb{R}^m , the distance between x and y is defined by $\|x - y\|$. In this case, one can define naturally the notation of a convergent sequence in \mathbb{R}^m as in the \mathbb{R} case. More precisely, we have the following definition.

Definition 6.1 A sequence (x_n) in \mathbb{R}^m is said to be convergent if there is an element $u \in \mathbb{R}^m$ such that $\lim_n \|x_n - u\| = 0$. Clearly such element is unique if it exists. In this case we call u the limit of (x_n) and write $u := \lim x_n$.

By using Eq 6.1, we have the following immediately.

Lemma 6.2 *Using the notation as above, a sequence (x_n) converges to u in \mathbb{R}^m if and only if the sequence $(x_n(k))$ converges to $u(k)$ in \mathbb{R} for all $k = 1, \dots, m$.*

Naturally, one can also have the following notation as in the \mathbb{R} case.

Definition 6.3 (i) *A point z in \mathbb{R}^m is said to be a limit point of a subset A of \mathbb{R}^m if for every $r > 0$, there is a point $a \in A$ such that $0 < \|z - a\| < r$. Also, A is said to be a closed set if A contains all its limit points.*

(ii) *A Cauchy sequence (x_n) in \mathbb{R}^m means that if for every $\varepsilon > 0$, there is a positive integer N such that $\|x_m - x_n\| < \varepsilon$ whenever $m, n \geq N$.*

Theorem 6.4 Cauchy Criterion: *A sequence in \mathbb{R}^m is convergent if and only if it is a Cauchy sequence.*

Proof: The Eq 6.1 tells us that if a sequence (x_n) in \mathbb{R}^m is a Cauchy sequence, then so is for each coordinate sequence $(x_n(k))$ in \mathbb{R} for $k = 1, \dots, m$. Hence the result is obtained immediately by using the Cauchy criterion for the real case. \square

In the rest of this section, we are going to show the Bolzano-Weierstrass Theorem in the higher dimensional case. Recall that a subsequence of (x_n) in \mathbb{R}^m is a sequence in \mathbb{R}^m given by a strictly increasing function $\phi : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$, i.e., it is $(x_{\phi(j)})_{j=1}^{\infty}$.

Theorem 6.5 Bolzano-Weierstrass Theorem: *Every bounded sequence in \mathbb{R}^m has a convergent subsequence.*

Proof: Let (x_n) be a bounded sequence in \mathbb{R}^m . Thanks to the Eq6.1 again, each coordinate sequence $(x_n(k))$ is also a bounded sequence of real numbers for $k = 1, \dots, m$. As $k = 1$, by the Bolzano-Weierstrass for the real sequence case, there is a convergent subsequence $(x_{\phi_1(j)}(1))_{j=1}^{\infty}$ of $(x_n(1))$, where $\phi_1 : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ is a strictly increasing function. As $k = 2$, we consider the subsequence $(x_{\phi_1(j)})$ of (x_n) . Using the Bolzano-Weierstrass for the real sequence case again, there is a convergent subsequence $(x_{\phi_2(j)}(2))_{j=1}^{\infty}$ of $(x_{\phi_1(j)}(2))$, where $\phi_2 : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ is a strictly increasing function. Next we consider the subsequence $(x_{\phi_2 \circ \phi_1(j)})$ of $(x_{\phi_1(j)})$, and so is the subsequence of (x_n) , for the case of $k = 3$. To repeat the same step, we get a subsequence $(x_{\phi_m \circ \dots \circ \phi_1(j)})$ of (x_n) so that each coordinate sequence $(x_{\phi_m \circ \dots \circ \phi_1(j)}(k))$ is convergent for all $k = 1, \dots, m$. Then by the Eq 6.1 again, the subsequence $(x_{\phi_m \circ \dots \circ \phi_1(j)})$ is convergent in \mathbb{R}^m as desired. \square

7 Limits of functions

Throughout this section let f be a real-valued function defined on a subset A of \mathbb{R} .

A point x_0 is called a *limit point* of A if for any $r > 0$, there is some element $a \in A$ such that $0 < |x_0 - a| < r$. We write $D(A)$ for the set of all limit points of A . Note that a limit point of A may not sit in A .

Definition 7.1 Let $c \in D(A)$. A number L is said to be a limit of f at c (note that $f(c)$ may not be defined!!) if for any ε , there is $\delta = \delta(\varepsilon) > 0$ (depends the choice of ε) such that

$$|f(x) - L| < \varepsilon \quad \text{whenever } x \in A \text{ and } 0 < |x - c| < \delta .$$

(Note: we only consider those points in A which are very close to c but do not equal to c !!!)

Remark 7.2 A number L is not a limit of f at c means if there is $\varepsilon > 0$ so that for any δ , we can find some $x' \in A$ with $|x' - c| < \delta$ but $|f(x') - L| \geq \varepsilon$.

Proposition 7.3 Using the notation as above if f has a limit at c , then its limit is unique. Consequently, if we write $\lim_{x \rightarrow c} f(x)$ for the limit of f at c , then this notation is well defined.

Proof: Let L' be a another limit of f at c . Let $\varepsilon > 0$. Then by the definition above, there are some positive numbers δ and δ' so that $|f(x) - L| < \varepsilon$ for any $x \in A$ with $0 < |x - c| < \delta$. Similarly, we have $|f(x) - L'| < \varepsilon$ for any $x \in A$ with $0 < |x - c| < \delta'$. Since $c \in D(A)$, we can find some $a \in A$ such that $0 < |c - a| < \delta''$, where $\delta'' = \min(\delta, \delta')$. This gives

$$|L - L'| \leq |L - f(a)| + |f(a) - L'| < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $L = L'$ as desired. □

Example 7.4 Let $A = (0, \infty)$. Define $f(x) := x^2 \sin \frac{1}{x}$.

(i) Show that $\lim_{x \rightarrow 0} f(x) = 0$.

In fact, it is noted that $|x^2| \leq |x|$ for all $x \in (0, 1)$. Let $\varepsilon > 0$. Thus, if we take $0 < \delta = \min(\varepsilon, 1)$, then we have

$$|f(x) - 0| \leq |x^2| \leq |x| < \varepsilon$$

whenever $x > 0$ with $|x - 0| < \delta$.

(ii) Using the ε - δ notation, show that $\lim_{x \rightarrow 0} f(x) \neq 1$.

Note that if we take $\varepsilon = 1/2$, then for any $\delta > 0$, we choose a positive integer N such that $0 < |\frac{1}{N\pi} - 0| < \delta$, and we have

$$|f(\frac{1}{N\pi}) - 1| = 1 > \varepsilon.$$

Therefore, 1 is not the limit of f at 0.

Proposition 7.5 Using the notation as above, let c be a limit point of A . Then the following are equivalent.

(i) $\lim_{x \rightarrow c} f(x)$ exists.

(ii) If (x_n) is a convergent sequence in $A \setminus \{c\}$ with $\lim x_n = c$, then the sequence $(f(x_n))$ is convergent.

In this case, $\lim_{x \rightarrow c} f(x) = \lim f(x_n)$ whenever a convergent sequence (x_n) in $A \setminus \{c\}$ with $\lim x_n = c$.

Proof: For showing (i) \Rightarrow (ii), we assume that $L = \lim_{x \rightarrow c} f(x)$ exists. Let (x_n) be a convergent sequence in $A \setminus \{c\}$ with the limit c . Let $\varepsilon > 0$. Then by the definition of the limit of a function, we can find $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in A$ with $0 < |x - c| < \delta$. On the other hand, since $\lim x_n = c$ and $x_n \neq c$ for all n , there is a positive integer N such that $0 < |x_n - c| < \delta$ for all $n \geq N$ and thus, $|f(x_n) - L| < \varepsilon$ for all $n \geq N$. Thus, the condition (ii) holds.

Suppose that the condition (ii) holds. We first claim that the sequence $(f(x_n))$ converges to the same limit whenever (x_n) is a convergent sequence in $A \setminus \{c\}$ such that $\lim x_n = c$. In fact, suppose that there are two sequences (x_n) and (y_n) in $A \setminus \{c\}$ with $\lim x_n = \lim y_n = c$ such that $u := \lim f(x_n)$ and $v := \lim f(y_n)$ are not the same. Put $w_n := x_k$ if $n = 2k$ and $w_n := y_k$ if $n = 2k + 1$ is odd. Then $\lim w_n = c$. The assumption (ii) implies that $\lim f(w_n)$ exists. However, $(f(x_n))$ and $(f(y_n))$ both are the subsequences of $(f(w_n))$ and they converge to the different limits. It leads to a contradiction.

Now we fix a sequence (x_n) in $A \setminus \{c\}$ such that $\lim x_n = c$. Then by the assumption $L := \lim f(x_n)$ exists. Suppose that L is not the limit of $f(x)$ at c . Thus, there is $\varepsilon > 0$ such that for any $\delta > 0$, we can find some $x' \in A$ with $0 < |x' - c| < \delta$ but $|f(x') - L| \geq \varepsilon$. From this, we see that for each positive integer n , there is $x'_n \in A$ with $0 < |x'_n - c| < 1/n$ but $|f(x'_n) - L| \geq \varepsilon$. Thus, the sequence (x'_n) sits in $A \setminus \{c\}$ and converges to c but L is not the limit of the sequence $(f(x'_n))$. This contradicts to the above observation. Therefore, the part (i) holds. \square

Proposition 7.5, together with Proposition 2.8, we have the following assertion immediately.

Proposition 7.6 *Let f and g be the functions defined on A . Let c be a limit point of A . Assume that $L := \lim_{x \rightarrow c} f(x)$ and $R := \lim_{x \rightarrow c} g(x)$ both exist. Then we have the following statements.*

1. $\lim_{x \rightarrow c} (f + g)(x)$ exists and $\lim_{x \rightarrow c} (f + g)(x) = L + R$
2. $\lim_{x \rightarrow c} (f \cdot g)(x)$ exists and $\lim_{x \rightarrow c} (f \cdot g)(x) = L \cdot R$.
3. if we further assume that $g(x) \neq 0$ for all $x \in A$ and $R \neq 0$, then $\lim_{x \rightarrow c} (f/g)(x)$ exists and $\lim_{x \rightarrow c} (f/g)(x) = L/R$.

The following result is regarded as the Cauchy criterion in the case of functions.

Proposition 7.7 *Using the notation as before, $\lim_{x \rightarrow c} f(x)$ exists if and only if for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x') - f(x'')| < \varepsilon$ whenever $x', x'' \in A$ with $0 < |x' - c| < \delta$ and $0 < |x'' - c| < \delta$.*

Proof: For showing (\Rightarrow) we assume that $L := \lim_{x \rightarrow c} f(x)$ exists. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that $|f(x) - L| < \varepsilon$ as $x \in A$ with $0 < |x - c| < \delta$. Thus, if $x', x'' \in A$ with $0 < |x' - c| < \delta$ and $0 < |x'' - c| < \delta$, we see that

$$|f(x') - f(x'')| \leq |f(x') - L| + |L - f(x'')| < 2\varepsilon.$$

Hence, the necessary condition holds.

Note that since c is a limit point of A , we can find a sequence Let (x_n) in $A \setminus \{c\}$ such that $\lim x_n = c$. Then the necessary condition implies that $(f(x_n))$ is a Cauchy sequence. In fact,

for any $\varepsilon > 0$, the necessary condition above gives $\delta > 0$ so that $|f(x') - f(x'')| < \varepsilon$ whenever $x', x'' \in A$ with $0 < |x' - c| < \delta$ and $0 < |x'' - c| < \delta$. Since $\lim x_n = c$, there is a positive integer N such that $|x_n - c| < \delta$ for all $n \geq N$ and hence, we have $|f(x_n) - f(x_m)| < \varepsilon$ for all $m, n \geq N$. Thus, $(f(x_n))$ is a Cauchy sequence. Hence, $\lim f(x_n)$ exists. The proof is complete by using Proposition 7.5.

□

Definition 7.8 Using the notation as before, let f be a function defined on A and let c be a limit point of A .

1. We say that f diverges to $+\infty$ (resp. $-\infty$) as x tends to c if for any $M > 0$, there is $\delta > 0$ such that $f(x) > M$ (resp. $f(x) < -M$) as $x \in A$ with $0 < |x - c| < \delta$. In this case, write $\lim_{x \rightarrow c} f(x) = +\infty$ (resp. $\lim_{x \rightarrow c} f(x) = -\infty$).
2. We further suppose that A is not bounded above. We say that f has a limit L as x tends to $+\infty$ if for any $\varepsilon > 0$, there is a positive number $R > 0$ such that $|f(x) - L| < \varepsilon$ as $x \in A$ with $x > R$. In this case, a limit must be unique if it exists. Write $\lim_{x \rightarrow \infty} f(x) = L$.
Similarly, one can define the notion $\lim_{x \rightarrow -\infty} f(x) = L$ when A is not bounded below. For simply, when we are talking about notion $\lim_{x \rightarrow \infty} f(x)$, A has been assumed to be unbounded above in advance.
3. Similarly, one can give a suitable definition for the notion: $\lim_{x \rightarrow \infty} f(x) = +\infty$.

Proposition 7.9 Using the notation as before, let f, g be the functions defined on A .

- (i) If $\lim_{x \rightarrow c} f(x) = +\infty$ and $\lim_{x \rightarrow c} g(x)$ exists, then $\lim_{x \rightarrow c} (f + g)(x) = +\infty$.
- (ii) If $\lim_{x \rightarrow c} f(x) = +\infty$ and $\lim_{x \rightarrow c} g(x) > 0$ exists, then $\lim_{x \rightarrow c} (f \cdot g)(x) = +\infty$.
- (iii) If $\lim_{x \rightarrow \infty} f(x) = +\infty$ and $\lim_{x \rightarrow \infty} g(x) > 0$ exists, then $\lim_{x \rightarrow \infty} (f \cdot g)(x) = +\infty$.

Proof: For showing part (ii), let $M > 0$. Since $l := \lim_{x \rightarrow c} g(x) > 0$, there is $\delta_1 > 0$ so that $g(x) > l - \frac{l}{2} = \frac{l}{2} > 0$ for all $x \in A$ with $0 < |x - c| < \delta_1$. Moreover, $\lim_{x \rightarrow c} f(x) = +\infty$, and so we can find $0 < \delta < \delta_1$ such that $f(x) > \frac{2M}{l}$ as $x \in A$ with $0 < |x - c| < \delta$ and hence in this case, we have

$$f(x)g(x) > \frac{2M}{l} \cdot \frac{l}{2} = M.$$

Part (iii) follows.

Using the similar argument, try to finish the proof by yourself. □

Remark 7.10 The assumption of the non-zero limits in Proposition 7.9(ii) and (iii) cannot be removed. For example, by considering $f(x) := 1/x; g(x) := x$ for $x > 0$, note that $\lim_{x \rightarrow 0} f(x) = \infty$ and $\lim_{x \rightarrow 0} g(x) = 0$ but $f(x)g(x) = 1$ for all $x > 0$.

Example 7.11 Let $p(x) := a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ be a polynomial of degree $n > 0$, where $x \in \mathbb{R}$. If the leading coefficient a_n of p is positive, then $\lim_{x \rightarrow +\infty} f(x) = +\infty$. In fact, since $a_n \neq 0$, we see that

$$p(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n} x^{-1} + \frac{a_{n-2}}{a_n} x^{-2} + \dots + \frac{a_0}{a_n} x^{-n} \right)$$

for all $x > 0$. In addition, since $a_n > 0$ and $n > 0$, clearly we have $\lim_{x \rightarrow +\infty} a_n x^n = +\infty$. The result follows immediately from Proposition 7.9.

Definition 7.12 A point c is called a right (resp. left) limit point of A if for any $r > 0$, there is some $x \in A$ such that $0 < x - c < r$ (resp. $0 < c - x < r$), i.e., $(c, c + r) \cap A \neq \emptyset$ (resp. $(c - r, c) \cap A \neq \emptyset$). Write $D_r(A)$ (resp. $D_l(A)$) for the set of right (resp. left) limit points of A .

Clearly, we have $D_r(A) \cup D_l(A) = D(A)$.

Example 7.13 We have the following examples.

1. If $A = (0, 1) \cup \{2\}$, then $D_r(A) = [0, 1)$ and $D_l(A) = (0, 1]$.
2. If $A = \{1, 1/2, 1/3, \dots\}$, then $D_r(A) = \{0\}$ and $D_l(A) = \emptyset$.

Definition 7.14 Using the notation as above, let $c \in D_r(A)$. We say that f has a right (resp. left) limit L of f at c if for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in A$ with $0 < x - c < \delta$ (resp. $0 < c - x < \delta$).

It is noted that if a right (resp. left) limit exists, then it is unique.

We write $\lim_{x \rightarrow c^+} f(x)$ and $\lim_{x \rightarrow c^-} f(x)$ for the right and left limit respectively.

Example 7.15 Let $A = \mathbb{R} \setminus \{0\}$. Define $f(x) = 1$ if $x > 0$; otherwise, $f(x) = -1$. Then $\lim_{x \rightarrow 0^+} f(x) = 1$ and $\lim_{x \rightarrow 0^-} f(x) = -1$. This function is called the sign function. We always denote it by $\text{sgn}(x)$.

Proposition 7.16 Let $c \in D_r(A) \cap D_l(A)$. Then $\lim_{x \rightarrow c} f(x)$ exists if and only if $\lim_{x \rightarrow c^+} f(x)$ and $\lim_{x \rightarrow c^-} f(x)$ both exist and $\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x)$. In this case, we have $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x)$.

Proposition 7.17 Let $f(x)$ be a function defined on $(0, \infty)$ and $g(x) = f(1/x)$. Then $\lim_{x \rightarrow +\infty} f(x)$ exists if and only if $\lim_{x \rightarrow 0^+} g(x)$ exists. In this case, we have $\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow 0^+} g(x)$.

8 Continuous functions

Throughout this section, let A be a non-empty subset of \mathbb{R} and let f be a function defined on A .

Definition 8.1 Let $c \in A$. We say that a function f is continuous at c if for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $x \in A$ with $|x - c| < \delta$. Furthermore, f is said to be continuous on A if it is continuous at every point in A .

Remark 8.2 Using the notation as above, note that

1. A function f is discontinuous at c if there is $\varepsilon > 0$ so that for any $\delta > 0$, we can find some $x \in A$ satisfying $|x - c| < \delta$ but $|f(x) - f(c)| \geq \varepsilon$.

2. If a point $c \in A$ is not a limit point of A , then a function f is continuous automatically at c . In fact, if $c \in A$ is not a limit point of A , then there is $r > 0$ such that $(c-r, c+r) \cap A = \{c\}$. Therefore, for any $\varepsilon > 0$, we can choose $\delta = r$ in the Definition 8.1 above.

Definition 8.3 Let A be subset of \mathbb{R} . A subset E of A is called an open subset of A if for each $c \in E$, there is $r > 0$ such that $(c-r, c+r) \cap A \subseteq E$.

Remark 8.4 Warning !!!!: Notice that if E an open subset of A , it does not imply that E is an open subset of \mathbb{R} .

For example, if we consider $A = [0, 1]$ and $E = (0, 1]$, then E is an open subset of A but it is not an open subset of \mathbb{R} .

The following result is directly obtained from the definition of a continuous function.

Proposition 8.5 Using the notation as above, then a function f is continuous on A if and only the pre-image $f^{-1}(V) := \{x \in A : f(x) \in V\}$ of any open subset V of \mathbb{R} is an open subset of A .

Proposition 8.6 Let $c \in A$. Then we have the following assertions.

- (i) If $c \in A$ is a limit point of A , then f is continuous at c if and only if $\lim_{x \rightarrow c} f(x) = f(c)$.
- (ii) f is continuous at c if and only if whenever a sequence (x_n) in A with $\lim x_n = c$, we have $\lim f(x_n) = f(c)$.

Proof: Part (i) follows directly from the Definition 8.1.

Part (ii) can be obtained by using a similar argument as in Proposition 7.6. Try to do it by yourself. □

Proposition 8.7 Let $c \in A$ and let f, g be functions defined on A . If f, g are continuous at c , then we have the following assertions.

- (i) The function $f + g$ is continuous at c .
- (ii) The product $f \cdot g$ is continuous at c .
- (iii) Moreover, if $g(x) \neq 0$ for all $x \in A$, then f/g is continuous at c .
- (iv) Moreover, if the image of f is contained in a subset B of \mathbb{R} and $h : B \rightarrow \mathbb{R}$ is continuous at $f(c)$, then the composition $h \circ f$ is continuous at c .

Proof: The above assertions follows immediately from Propositions 2.8 and 8.6. Alternatively, they can be shown directly by the definition.

For showing part (ii), since g is continuous at c , there is $\delta_1 > 0$ such that $|f(x) - f(c)| < 1$ and hence, $|f(x)| < 1 + |f(c)|$ for all $x \in A$ with $|x - c| < \delta_1$. Using the continuity of f and g at c , there exists $0 < \delta < \delta_1$ so that $|f(x) - f(c)| < \varepsilon$ and $|g(x) - g(c)| < \varepsilon$ as $x \in A$ and $|x - c| < \delta$. Therefore, we have

$$|f(x)g(x) - f(c)g(c)| \leq |f(x)g(x) - f(c)g(x)| + |f(c)g(x) - f(c)g(c)| \leq \varepsilon(1 + |g(c)| + |f(c)|)$$

as $x \in A$ and $|x - c| < \delta$. Part (ii) follows.

By using part (ii), we need to show that the function $1/g(x)$ is continuous at c . Note that we may assume $g(c) > 0$ (otherwise by considering $-g(x)$). $g(x)$ is continuous at c , and so there is $\delta_1 > 0$ so that $|g(x) - g(c)| < g(c)/2$ and hence, $g(x) > g(c)/2$ for all $x \in A$ and $|x - c| < \delta_1$. Now let $\varepsilon > 0$, there is $0 < \delta < \delta_1$ so that $|g(x) - g(c)| < \delta$ as $x \in A$ and $|x - c| < \delta$. Therefore, we have

$$\left| \frac{1}{g(x)} - \frac{1}{g(c)} \right| = \frac{|g(x) - g(c)|}{g(x)g(c)} \leq \frac{2\varepsilon}{g(c)^2}$$

for all $x \in A$ with $|x - c| < \delta$. The proof of (iii) is complete.

The last assertion follows clearly from the definition. □

Before showing the following important result, we first recall that a subset A is said to be compact if for any sequence (x_n) in A has a convergent subsequence (x_{n_k}) such that $\lim_k x_{n_k} \in A$. Moreover, A is compact if and only if it is a closed and bounded set.

Theorem 8.8 *If f is a continuous function defined on a compact set A , then f is a bounded function. Moreover, there are x_1 and x_2 in A such that $f(x_1) = \min\{f(x) : x \in A\}$ and $f(x_2) = \max\{f(x) : x \in A\}$.*

Proof: First, we show that the set $f(A)$ is bounded above. We give two different methods about the claim.

Method I:

Suppose that f is not bounded above. Then for each positive integer n , there is $x_n \in A$ such that $f(x_n) \geq n$. A is compact, and so there is a convergent subsequence (x_{n_k}) with $c := \lim x_{n_k} \in A$. Note that since f is continuous at c , we see that the sequence $(f(x_{n_k}))$ converges to $f(c)$ and thus, $(f(x_{n_k}))$ is a bounded sequence but $f(x_{n_k}) \geq n_k$ for all k . It leads to a contradiction because $n_k \rightarrow \infty$ as $k \rightarrow \infty$.

Method II:

Notice that since f is continuous on A , then for each element $x \in A$, there is $\delta(x) > 0$ such that $f(u) < f(x) + 1$ as $u \in A$ with $|u - x| < \delta(x)$. Note that if we put $J_x := (x - \delta(x), x + \delta(x))$ for $x \in A$, then we have $A \subseteq \bigcup_{x \in A} J_x$. Using the compactness of A , there are finitely many $x_1, \dots, x_N \in A$ such that $A \subseteq J_{x_1} \cup \dots \cup J_{x_N}$. Then implies that $f(x) \leq \max(1 + f(x_1), \dots, 1 + f(x_N))$ for all $x \in A$, so the set $f(A)$ is bounded above.

Next, we want to show that $f(a) = \max\{f(x) : x \in A\}$ for some $a \in A$.

In fact, note that since the set $\{f(x) : x \in A\}$ is bounded above, $L := \sup\{f(x) : x \in A\}$ exists. Thus, there exists a sequence (x_n) in A such that $\lim f(x_n) = L$. Using the compactness of A , there is a convergent subsequence (x_{n_k}) of (x_n) with $a := \lim x_{n_k} \in A$. Thus, we have $f(a) = \lim f(x_{n_k})$ and thus, $f(a) = L$ as desired.

By considering $-f$, we get $f(x_1) = \min\{f(x) : x \in A\}$ for some $x_1 \in A$. The proof is complete. □

Remark 8.9 The assumption of compactness in Theorem 8.8 cannot be removed.

For example if $A = [1, \infty)$ and $f(x) = 1/x$ for $x \in A$, then there is no points attains its minimum on A although f is a bounded function.

Theorem 8.10 *If f is a continuous function defined on a compact set, then the image $f(A) := \{f(x) : x \in A\}$ is compact.*

Proof: Method I:

It suffices to show that $f(A)$ is a closed and bounded set. We have shown that $f(A)$ is bounded by Theorem 8.8. We need to show that $f(A)$ is closed. By applying Proposition 4.7, we need to claim that if (x_n) is a sequence in A so that $(f(x_n))$ is convergent, then the limit $L := \lim f(x_n) \in f(A)$. Indeed, by the compactness of A , (x_n) has a convergent subsequence (x_{n_k}) such that $c := \lim x_{n_k} \in A$. f is continuous at c , and so $\lim f(x_{n_k}) = f(c)$ and thus, $L = f(c) \in f(A)$ as required.

Method II:

Let $\{J_i\}_{i \in I}$ be an open cover of $f(A)$. Since f is continuous on A , Proposition 8.5 implies that for each element $a \in A$, there are $\delta_a > 0$ and $i_a \in I$ such that $f((a - \delta_a, a + \delta_a) \cap A) \subseteq J_{i_a}$. Note that we have

$$A \subseteq \bigcup_{a \in A} (a - \delta_a, a + \delta_a).$$

Then by the compactness of A , there are finitely many a_1, \dots, a_N in A such that

$$A \subseteq \bigcup_{k=1}^N (a_k - \delta_{a_k}, a_k + \delta_{a_k}).$$

Therefore, we have

$$f(A) \subseteq \bigcup_{k=1}^N f((a_k - \delta_{a_k}, a_k + \delta_{a_k}) \cap A) \subseteq \bigcup_{k=1}^N J_{i_{a_k}}.$$

□

Remark 8.11 In general, the image of a closed set under a continuous map is not necessarily closed. For example, $A = [1, \infty)$ and $f(x) = 1/x, x \in A$. Note that A is a closed set but $f(A) = (0, 1]$ is not closed.

Definition 8.12 *Two subsets A and B are said to be homeomorphic if there is a bijection f from A onto B such that f and the inverse f^{-1} both are continuous. In this case, f is called a homeomorphism.*

Proposition 8.13 *Suppose that A and B are homeomorphic. If A is compact, then so is B .*

Proof: It can be shown directly by Theorem 8.10. □

Example 8.14 *By applying Theorem 8.10, it is impossible to find a continuous surjection from $[0, 1]$ onto $[0, 1)$ because $[0, 1]$ is compact but $[0, 1)$ is not. Therefore, $[0, 1]$ is not homeomorphic to $[0, 1)$.*

Proposition 8.15 *Let A and B be non-empty subsets of \mathbb{R} . Let $f : A \rightarrow B$ be a continuous bijection. If A is compact, then f is a homeomorphism, i.e., the inverse f^{-1} is continuous.*

Proof: Put $y = f(x)$ and $g(y) = f^{-1}(x)$, $x \in A$. Suppose that the function g is discontinuous at some $b \in B$. Then, there is $\varepsilon > 0$ so that for any $\delta > 0$, there is $y \in B$ so that $|y - b| < \delta$ but $|g(y) - g(b)| \geq \varepsilon$. By considering $\delta = 1/n$ for $n = 1, 2, \dots$. Therefore, there is a sequence (y_n) in B so that $\lim y_n = b$ and $|g(y_n) - g(b)| \geq \varepsilon$ for all n . Let $x_n = g(y_n) \in A$. Then by the compactness of A , (x_n) has a convergent subsequence (x_{n_k}) such that $a := \lim x_{n_k} \in A$. Note that $b = \lim y_{n_k} = \lim f(x_{n_k}) = f(a)$ because f is continuous and $\lim y_n = b$. Thus, $a = g(b)$. Therefore, we have $\lim g(y_{n_k}) = \lim x_{n_k} = a = g(b)$ which leads to a contradiction because $|g(y_n) - g(b)| \geq \varepsilon$ for all n . \square

Remark 8.16 The assumption of compactness of the domain on Proposition 8.15 cannot be removed. For example, by considering $A = [0, 1) \cup [1, 2]$ and $B = [0, 2]$, a function $f : A \rightarrow B$ is defined by $f(x) = x$ for $x \in [0, 1)$ and $f(x) = x - 1$ for $x \in [1, 2]$. Then f is a continuous bijection but its inverse is discontinuous at $y = 1$. Note that A is non-compact in this case.

Theorem 8.17 Intermediate Value Theorem *Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Assume that $f(a) < L < f(b)$. Then there is $c \in (a, b)$ such that $f(c) = L$.*

Proof: If we consider the function $x \in [a, b] \mapsto f(x) - L$, then we may assume that $L = 0$, i.e., $f(a) < 0 < f(b)$. We want to show that there is $c \in (a, b)$ so that $f(c) = 0$.

Method 1:

Let $S := \{x \in [a, b] : f(x) > 0\}$. Note that S is non-empty a bounded below set since $b \in S$ and $x > b$ for all $x \in S$. Thus, $c := \inf S$ exists. We will show that $f(c) = 0$. Note that for each positive integer n , there is $x_n \in S$ satisfying $c \leq x_n < c + 1/n$, and so $\lim x_n = c$. Since $a \leq x_n \leq b$ for all n , we see that $c \in [a, b]$. By the continuity of f and $f(x_n) > 0$ for all n , we have $\lim f(x_n) = f(c)$ and $f(c) \geq 0$. We want to show that it is impossible if $f(c) > 0$. Note that $c > a$ since $f(a) < 0$. Therefore, there is $\delta > 0$ such that $a < c - \delta$ and $|f(x) - f(c)| < f(c)/2$ as $x \in [a, b]$ with $|x - c| < \delta$. Thus, if we fix a point x_1 such that $a < c - \delta < x_1 < c \leq b$, then we have $f(x_1) > f(c)/2 > 0$. This implies that $x_1 \in S$ and $x_1 < c$. It is a contradiction because c is a lower bound for the set S . Therefore, $f(c) = 0$.

Method 2:

Let $[a_1, b_1] = [a, b]$. We want to construct inductively a sequence of closed and bounded intervals $\{[a_k, b_k]\}_{k=1}^n$, where $1 \leq n \leq +\infty$, satisfying the following conditions.

1. $[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots$.
2. $b_k - a_k = \frac{1}{2}(b_{k-1} - a_{k-1})$, for all $2 \leq k \leq n$.
3. $f(a_k) < 0 < f(b_k)$, for all $1 \leq k \leq n$.

Suppose that the sequence of closed and bounded intervals $([a_k, b_k])$ has been constructed for $1 \leq k \leq n$. We want to construct $[a_{n+1}, b_{n+1}]$ so that it satisfies the conditions (1) – (3) above. Put $m_n := \frac{a_n + b_n}{2}$. If $f(m_n) = 0$, then the result follows. Otherwise, if $f(m_n) > 0$, then we put $[a_{n+1}, b_{n+1}] = [a_n, m_n]$. If $f(m_n) < 0$, then we put $[a_{n+1}, b_{n+1}] = [m_n, b_n]$. Therefore, if $f(m_n) \neq 0$ for all $n = 1, 2, \dots$, then we have an infinite sequence of $([a_k, b_k])$ satisfying the conditions (1) – (3) above. By applying the Nested Intervals Theorem in this case, we have $\bigcap_{k=1}^{\infty} [a_k, b_k] = \{c\}$ for some $c \in [a, b]$. Note that we have $\lim a_k = \lim b_k = c$. f is continuous at c , so we have $f(c) = \lim f(a_k) = \lim f(b_k)$. From this, together with the condition (3) above, we have $f(c) = 0$. The proof is complete. \square

Recall that an interval is a non-empty subset of \mathbb{R} which is one of the following forms.

1. (Bounded case): $[a, b]$; $[a, b)$; $(a, b]$ and (a, b) for $a < b$.
2. (Unbounded case): $[a, +\infty)$; $(a, +\infty)$; $(-\infty, b]$; $(-\infty, b)$ and \mathbb{R} .

Proposition 8.18 *Let A be subset of \mathbb{R} . Assume that A has at least two points. Then the followings are equivalent.*

1. A is an interval.
2. For each pair of elements $a, b \in A$ with $a < b$, we have $[a, b] \subseteq A$.

Proof: (1) \Rightarrow (2) is clear. We want to show (2) \Rightarrow (1). Assume that the condition (2) holds.

First, we assume that A is bounded. Then $L := \sup A$ and $l := \inf A$ both exist. Then $x \in [l, L]$ for any $x \in A$, so $A \subseteq [l, L]$. Now, if L and l are in A , then the condition (2) implies that $[l, L] \subseteq A$ and thus, $A = [l, L]$. By using the similar argument for the other cases, i.e., $l \in A$ and $L \notin A$; $l \notin A$ and $L \in A$; $l \notin A$ and $L \notin A$, we see that A is equal to $[l, L)$; $(l, L]$ and (l, L) respectively.

Similarly, the result can be obtained in the unbounded case. □

Theorem 8.19 *Let f be a continuous function defined on A . If A is an interval, then so is its image $f(A)$.*

Proof: By using Proposition 8.18, we need to show that $[c, d] \subseteq f(A)$ whenever $c, d \in f(A)$ with $c < d$. In fact, let $f(a) = c$ and $f(b) = d$ for some $a, b \in A$. We may assume that $a < b$. Note that since A is an interval, we have $[a, b] \subseteq A$. By applying the Intermediate Value Theorem, for any element $L \in [c, d]$, there is an element x_1 between a and b such that $L = f(x_1) \in f(A)$, and hence $[c, d] \subseteq f(A)$. The proof is complete. □

Example 8.20 *By Theorem 8.19, there is no continuous surjections from $[0, 1]$ onto $[0, 1] \cup [2, 3]$. Hence, the set $[0, 1]$ is not homeomorphic to $[0, 1] \cup [2, 3]$.*

9 Uniform continuous functions

Throughout this section, let f be a function defined on a non-empty subset of \mathbb{R} .

Definition 9.1 *A function f is said to be uniformly continuous on A if for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in A$ with $|x - y| < \delta$.*

Remark 9.2 *A function f is not uniformly continuous on A if there is $\varepsilon > 0$ such that for any $\delta > 0$, there are $x, y \in A$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$.*

Example 9.3 (i) Let $f(x) = x^2$ for $x \in [0, \infty)$. Then f is not uniformly continuous on $[0, \infty)$. In fact for any positive integer n , we have

$$\left|f\left(n + \frac{1}{n}\right) - f(n)\right| = (2n + 1/n)(1/n) = 2 + \frac{1}{n^2} \geq 2.$$

Therefore, if we let $\varepsilon = 2$, then for any $\delta > 0$, we choose a positive integer n such that $1/n < \delta$, so $n + 1/n$ and n in $[0, \infty)$ with $|n + 1/n - n| < \delta$ but $|f(n + \frac{1}{n}) - f(n)| \geq 2$. Thus, f is not uniformly continuous on $[0, \infty)$.

Note that from this example we see that a continuous function need not be uniformly continuous on its domain.

(ii) Let $f(x) = x^2$ for $x \in [0, 1]$. Then f is uniformly continuous on $[0, 1]$. In fact for $x, y \in [0, 1]$ we have

$$|f(x) - f(y)| = |x - y||x + y| \leq 2|x - y|.$$

Let $\varepsilon > 0$. Then we can choose $0 < \delta < \varepsilon/2$, so we have $|f(x) - f(y)| \leq 2|x - y| < \varepsilon$ whenever $x, y \in [0, 1]$ with $|x - y| < \delta$. Thus, f is uniformly continuous on $[0, 1]$.

Theorem 9.4 Let f be a continuous function on A . If A is compact, then f is uniformly continuous on A .

Proof: Method I:

Suppose that A is compact but f is not uniformly continuous on A . Then there is $\varepsilon > 0$ such that for any $\delta > 0$, there are $x, y \in A$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$. Consider $\delta = 1/n$ for $n = 1, 2, \dots$. Then for any positive integer n , there are x_n and y_n such that $|x_n - y_n| < 1/n$ but $|f(x_n) - f(y_n)| \geq \varepsilon$.

Then by the compactness of A , the sequence (x_n) has a convergent subsequence (x_{n_k}) such that $a := \lim_k x_{n_k}$. By applying the compactness of A , the sequence (y_{n_k}) has a convergent subsequence $(y_{n_{k_i}})$ with $b := \lim_i y_{n_{k_i}} \in A$. Note that we still have $a := \lim_i x_{n_{k_i}}$. Since $|x_{n_{k_i}} - y_{n_{k_i}}| < 1/n_{k_i}$ for all $i = 1, 2, \dots$, we have $a = b$. Hence, we have

$$\lim_i f(x_{n_{k_i}}) = f(a) = f(b) = \lim_i f(y_{n_{k_i}}),$$

and so we have

$$0 < \varepsilon \leq |f(x_{n_{k_i}}) - f(y_{n_{k_i}})| \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

It leads to a contradiction.

Method II:

Let $\varepsilon > 0$. Let $a \in A$. Since f is continuous at a , there is $\delta_a > 0$ so that $|f(x) - f(a)| < \varepsilon$ as $x \in A$ and $|x - a| < \delta_a$. Put $J_a := (a - \frac{\delta_a}{2}, a + \frac{\delta_a}{2})$. Then we have $A \subseteq \bigcup_{a \in A} J_a$. By using

the compactness of A , there are finitely many $a_1, \dots, a_N \in A$ such that $A \subseteq \bigcup_{k=1}^N J_{a_k}$. Take

$0 < \delta < \frac{1}{2}\delta_{a_k}$ for all $k = 1, \dots, N$. Let $x, x' \in A$ with $|x - x'| < \delta$. Note that $x \in J_{a_i}$ for some $1 \leq i \leq N$. Then we have $|x - a_i| < \frac{1}{2}\delta_{a_i} < \delta_{a_i}$ and $|x' - a_i| \leq |x' - x| + |x - a_i| < \delta_{a_i}$. Thus, we have

$$|f(x) - f(x')| \leq |f(x) - f(a_i)| + |f(a_i) - f(x')| < 2\varepsilon.$$

The proof is complete. □

Theorem 9.5 Weierstrass approximation theorem *Let $f(x)$ be a real-valued continuous function defined on $[0, 1]$. Then for any $\varepsilon > 0$, there is a polynomial function $p(x)$ such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [0, 1]$.*

Proof: Before showing the result, let us recall a class of polynomial functions in the following, called *Bernstein polynomials*. For each positive integer n , set

$$B_n(x) := \sum_{k=0}^n a_k \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{where } a_k \in \mathbb{R}$$

for $x \in [0, 1]$. Using the binomial theorem, clearly we have

$$\sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = 1 \tag{9.1}$$

for all $x \in [0, 1]$. Taking $x(1-x) \frac{d}{dx}$ in Eq 9.1, we have

$$\sum_{k=0}^n \binom{n}{k} [x^k (1-x)^{n-k}] (k - nx) = 0.$$

Again taking $\frac{1}{n^2} x(1-x) \frac{d}{dx}$ on both side and applying the product rule of derivatives, we have

$$\sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \left(x - \frac{k}{n}\right)^2 = \frac{x(1-x)}{n}.$$

This gives

$$\sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \left(x - \frac{k}{n}\right)^2 \leq \frac{1}{4n} \quad \text{for all } x \in [0, 1] \tag{9.2}$$

since $\max\{x(1-x) : x \in [0, 1]\} = 1/4$.

Now let

$$p_n(x) := \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \quad \text{for } x \in [0, 1].$$

The theorem is obtained if we show that for any $\varepsilon > 0$, there is a positive integer N such that $|f(x) - p_n(x)| < \varepsilon$ for all $n \geq N$ and for all $x \in [0, 1]$. Note that using the Eq9.1, we see that

$$|f(x) - p_n(x)| \leq \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} |f(x) - f\left(\frac{k}{n}\right)| \tag{9.3}$$

for all $x \in [0, 1]$ and for all $n = 1, 2, \dots$. On the other hand, since f is continuous on $[0, 1]$, f is uniformly continuous on $[0, 1]$. Thus, for any $\varepsilon > 0$, there is $\delta > 0$ such that $|f(u) - f(v)| < \varepsilon$ whenever $u, v \in [0, 1]$ with $|u - v| < \delta$. Now for any $x \in [0, 1]$ and $n = 1, 2, \dots$, we put

$$A := \sum_{k: |x - \frac{k}{n}| < \delta} \binom{n}{k} x^k (1-x)^{n-k} |f(x) - f\left(\frac{k}{n}\right)|$$

and

$$B := \sum_{k: |x - \frac{k}{n}| \geq \delta} \binom{n}{k} x^k (1-x)^{n-k} |f(x) - f\left(\frac{k}{n}\right)|.$$

Then the Eq 9.3 can be written as the following

$$|f(x) - p_n(x)| \leq A + B.$$

Notice that by the choice of δ , we have

$$A \leq \varepsilon \sum_{k: |x - \frac{k}{n}| < \delta} \binom{n}{k} x^k (1-x)^{n-k} = \varepsilon.$$

We are now going to estimate the value B . By using Eq 9.2, we have

$$\delta^2 \sum_{k: |x - \frac{k}{n}| \geq \delta} \binom{n}{k} x^k (1-x)^{n-k} \leq \sum_{k: |x - \frac{k}{n}| \geq \delta} \binom{n}{k} x^k (1-x)^{n-k} \left(x - \frac{k}{n}\right)^2 \leq \frac{1}{4n}.$$

This gives

$$\sum_{k: |x - \frac{k}{n}| \geq \delta} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{1}{4n\delta^2}.$$

Since f is continuous, f is bounded on $[0, 1]$. If we put $M := \sup |f(x)|$, then we can now conclude that

$$B \leq 2M \sum_{k: |x - \frac{k}{n}| \geq \delta} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{M}{2n\delta^2}$$

for all $x \in [0, 1]$ and for all $n = 1, 2, \dots$. Therefore, if we choose a positive integer N so that $\frac{M}{2n\delta^2} < \varepsilon$ for all $n \geq N$, then $|f(x) - p_n(x)| \leq A + B < 2\varepsilon$ for all $n \geq N$ and for all $x \in [0, 1]$ as desired. The proof is complete. \square

Definition 9.6 Let A be a non-empty subset of \mathbb{R} . A function $f : A \rightarrow \mathbb{R}$ is called a Lipschitz function if there is a constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. In this case.

Furthermore, if we can find such $0 < C < 1$, then we call f a contraction.

Clearly we have the following property.

Proposition 9.7 Every Lipschitz function is uniformly continuous on its domain.

Example 9.8 (i) : The sine function $f(x) = \sin x$ is a Lipschitz function on \mathbb{R} since we always have $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

(ii) : Define a function f on $[0, 1]$ by $f(x) = x \sin(1/x)$ for $x \in (0, 1]$ and $f(0) = 0$. Then f is continuous on $[0, 1]$ and thus f is uniformly continuous on $[0, 1]$, but note that f is not a Lipschitz function. In fact, for any $C > 0$, if we consider $x_n = \frac{1}{2n\pi + (\pi/2)}$ and $y_n = \frac{1}{2n\pi}$, then $|f(x_n) - f(y_n)| > C|x_n - y_n|$ if and only if

$$\frac{2}{\pi} \cdot \frac{(2n\pi + \frac{\pi}{2})(2n\pi)}{2n\pi + \frac{\pi}{2}} = 4n > C.$$

Therefore, for any $C > 0$, there are $x, y \in [0, 1]$ such that $|f(x) - f(y)| > C|x - y|$ and hence f is not a Lipschitz function on $[0, 1]$.

Proposition 9.9 Let A be a non-empty closed subset of \mathbb{R} . If $f : A \rightarrow A$ is a contraction, then there is a unique fixed point of f , i.e., there is a point $a \in A$ such that $f(a) = a$.

Proof: First we show the existence. f is a contraction on A , so there is $0 < C < 1$ such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in A$. Fix $x_1 \in A$. Since $f(A) \subseteq A$, we can inductively define a sequence (x_n) in A by $x_{n+1} = f(x_n)$ for $n = 1, 2, \dots$. Note that we have

$$|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq C|x_n - x_{n-1}|$$

for all $n = 2, 3, \dots$. This gives

$$|x_{n+1} - x_n| \leq C^{n-1}|x_2 - x_1|$$

for $n = 2, 3, \dots$. Thus, for any $n, p = 1, 2, \dots$, we see that

$$|x_{n+p} - x_n| \leq \sum_{i=n}^{n+p-1} |x_{i+1} - x_i| \leq |x_2 - x_1| \sum_{i=n}^{n+p-1} C^{i-1}.$$

Since $0 < C < 1$, for any $\varepsilon > 0$, there is N such that $\sum_{i=n}^{n+p-1} C^{i-1} < \varepsilon$ for all $n \geq N$ and $p = 1, 2, \dots$. Therefore, (x_n) is a Cauchy sequence and thus the limit $a := \lim_n x_n$ exists. A is closed, so we have $a \in A$ and hence f is continuous at a . On the other hand, since $x_{n+1} = f(x_n)$, we have $a = f(a)$ by taking $n \rightarrow \infty$.

Finally, we show the uniqueness of the fixed point. In fact, if a and b are the fixed points of f and $a \neq b$, then we have $|a - b| = |f(a) - f(b)| \leq C|a - b| < |a - b|$ because $0 < C < 1$. It leads to a contradiction. The proof is complete. \square

Remark 9.10 Proposition 9.9 does not hold if f is not a contraction. For example, if we consider $f(x) = x - 1$ for $x \in \mathbb{R}$, clearly we have $|f(x) - f(y)| = |x - y|$ and f has no fixed point in \mathbb{R} .

Proposition 9.11 Let f be a continuous function defined on (a, b) . The the followings are equivalent.

- (i) There exists a continuous function $F : [a, b] \rightarrow \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in (a, b)$.
- (ii) f is uniformly continuous on (a, b) .
- (iii) The limits $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow b^-} f(x)$ both exist.

In this case, this continuous extension F is uniquely determined by f . In fact, $F(a) = \lim_{x \rightarrow a^+} f(x)$ and $F(b) = \lim_{x \rightarrow b^-} f(x)$.

Proof: For (i) \Rightarrow (ii), we assume that (i) holds. Then by Theorem 9.4, F is uniformly continuous on $[a, b]$, so $f = F|_{(a, b)}$ is uniformly continuous on (a, b) .

For (ii) \Rightarrow (iii), we are going to show that $\lim_{x \rightarrow b^-} f(x)$ exists.

It suffices to show that the sequence $(f(x_n))$ converges to the same limit whenever any sequence (x_n) in (a, b) that converges to b .

First, we claim that $(f(x_n))$ is a Cauchy sequence for any such sequence (x_n) in (a, b) . Let $\varepsilon > 0$. Then by the assumption (ii), there is $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ as

$x, y \in (a, b)$ with $|x - y| < \delta$. Now since $\lim x_n = b$ and thus (x_n) is a Cauchy sequence. Therefore, we can find a positive N such that $|x_m - x_n| < \delta$ when $m, n \geq N$. This gives $|f(x_m) - f(x_n)| < \varepsilon$ as $m, n \geq N$. The claim follows and thus, the limit $\lim_{n \rightarrow \infty} f(x_n)$ exists. Next we want to show that if (x_n) and (y_n) both are the sequences in (a, b) that converge to b , then $\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} f(y_n)$. Let $L = \lim_{n \rightarrow \infty} f(x_n)$ and $L' = \lim_{n \rightarrow \infty} f(y_n)$. Let $\varepsilon > 0$ and let δ be given by the uniform continuity of f . Since $\lim x_n = \lim y_n$, we can choose a positive integer N large enough so that $|x_N - y_N| < \delta$. In addition, such N satisfies $|f(x_N) - L| < \varepsilon$ and $|f(y_N) - L'| < \varepsilon$ because $L = \lim_{n \rightarrow \infty} f(x_n)$ and $L' = \lim_{n \rightarrow \infty} f(y_n)$. This implies that

$$|L - L'| \leq |L - f(x_N)| + |f(x_N) - f(y_N)| + |f(y_N) - L'| < 3\varepsilon$$

for all $\varepsilon > 0$. Thus, $L = L'$ and hence, the limit $\lim_{x \rightarrow b^-} f(x)$ exists.

The proof of the case $\lim_{x \rightarrow a^+} f(x)$ is similar.

Finally, we show $(iii) \Rightarrow (i)$. Define $F(a) := \lim_{x \rightarrow a^+} f(x)$; $F(b) := \lim_{x \rightarrow b^-} f(x)$ and $F(x) := f(x)$ for $x \in (a, b)$. Note that F is continuous on $[a, b]$. In fact, we have $F(a) = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} F(x)$ and $F(b) = \lim_{x \rightarrow b^-} f(x) = \lim_{x \rightarrow b^-} F(x)$. Thus, F is continuous at $x = a$ and b .

The last assertion follows immediately from the continuity of F . The proof is complete. \square

Remark 9.12 Indeed, in the proof of Proposition 9.11 $(i) \Rightarrow (ii)$ above, we have shown the following fact. Suppose that f is uniformly continuous function defined on A . If (x_n) is a Cauchy sequence in A , then so is the sequence $(f(x_n))$. We can use this simple observation to see a function "NOT" being uniformly continuous on its domain.

Note the assumption of the uniform continuity of f is essential in here by considering the simple example that $f(x) = \frac{1}{x}$, $x \in A := (0, 1]$ and $x_n = \frac{1}{n}$, $n = 1, 2, \dots$

Definition 9.13 A function $s : [a, b] \rightarrow \mathbb{R}$ is called a step function (resp. piecewise linear) if there exist finitely many points $a = x_0 < x_1 < \dots < x_n = b$ such that s is a constant on each (x_{k-1}, x_k) (resp. linear on $[x_{k-1}, x_k]$, i.e. $s(x) = m_k x + b_k$) for all $k = 1, \dots, n$.

Proposition 9.14 If f is a continuous function defined on a closed and bounded interval $[a, b]$, then it can be uniformly approximated by step functions (resp. piecewise linear functions), that is, for each $\varepsilon > 0$, there exists a step function s (resp. piecewise linear function) defined on $[a, b]$ such that $|f(x) - s(x)| < \varepsilon$ for all $x \in [a, b]$.

Proof: By using Theorem 9.4, we first note that f is uniformly continuous on $[a, b]$. Let $\varepsilon > 0$. Then there is $\delta > 0$ so that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in (a, b)$ with $|x - y| < \delta$. If we choose a partition $a = x_0 < \dots < x_n = b$ on $[a, b]$ such that $|x_k - x_{k-1}| < \delta$ for $k = 1, \dots, n$. Now if we let $s(x) := f(x_{k-1})$ when $x \in [x_{k-1}, x_k)$, then s is the step function as desired. Using the similar argument, the result is obtained for the case of piecewise linear functions. \square

10 Monotone Functions

Using the notation given as before, f is a function defined on a subset A of \mathbb{R} . f is called a *monotone function* if it is either increasing or decreasing. The following results also hold for decreasing functions by considering $-f$ instead. Recall that c is a *right (resp. left) limit point* of A if for any $r > 0$ we have $(c, c+r) \cap A \neq \emptyset$ (resp. $(c-r, c) \cap A \neq \emptyset$).

Proposition 10.1 *Let f be an increasing function on A . Let $c \in A$. Put*

$$L(c) := \inf\{f(x) : x \in A, x > c\} \quad \text{if} \quad \{x \in A, x > c\} \neq \emptyset.$$

Similarly, we put

$$l(c) := \sup\{f(x) : x \in A, x < c\} \quad \text{if} \quad \{x \in A : x < c\} \neq \emptyset.$$

If c is a right (resp. left) limit point of A , then $L(c) = f(c+) := \lim_{x \rightarrow c+} f(x)$ (resp. $l(c) = f(c-) := \lim_{x \rightarrow c-} f(x)$).

Proof: First, we want to prove that if c is a right limit point of A , then the right limit $f(c+)$ exists. Since c is a right limit point of A , $\{f(x) : x \in A, x > c\} \neq \emptyset$. f is increasing, so $f(c)$ is a lower bound of the set $\{f(x) : x \in A, x > c\}$. The Axiom of Completeness implies that $L(c) := \inf\{f(x) : x \in A, x > c\}$ exists and $f(c) \leq L(c)$. Thus, for any $\varepsilon > 0$, there is $x_1 \in A$ with $x_1 > c$ such that $f(x_1) < L(c) + \varepsilon$. Hence, if we take $0 < \delta < x_1 - c$, then $L(c) - \varepsilon < L(c) \leq f(x) \leq f(x_1) < L(c) + \varepsilon$ whenever $x \in (c, c + \delta)$. Thus, $L(c) = f(c+)$ as desired.

The proof for the case of a left limit point is similar. □

Proposition 10.2 *Using the notation given as in Proposition 10.1, let f be a strictly increasing (not necessary continuous) function defined on an interval I , i.e. $f(x_1) < f(x_2)$ if and only if $x_1 < x_2$ as $x_1, x_2 \in I$. Let $g : f(I) \rightarrow I$ be the inverse of f . If $d \in f(I)$, then $g(d) = L(d) := \inf\{g(y) : y \in f(I), y > d\}$ (resp. $g(d) = l(d) := \sup\{g(y) : y \in f(I), y > d\}$) provided $L(d)$ (resp. $l(d)$) exists.*

In addition, if d is a right (resp. left) limit point of $f(I)$, then $g(d) = g(d+)$ (resp. $g(d) = g(d-)$).

Consequently, the inverse function $g : f(I) \rightarrow I$ is continuous.

Proof:

Note that g is also strictly increasing on $f(I)$. Let $c := g(d)$, hence $c \in I$ and $f(c) = d$. g is increasing, so $g(d) \leq L(d)$ whenever $L(d)$ exists. We now suppose that $g(d) < L(d)$, thus we can choose a point z such that $c = g(d) < z < L(d)$. Then by the definition of $L(d)$, there is $y_1 \in f(I)$ with $y_1 > d$. Thus, we have $z < L(d) \leq g(y_1)$. If we let $x_1 = g(y_1)$, then $x_1 \in I$ and $c < z < L(d) \leq x_1$. I is an interval, so $z \in (c, x_1) \subseteq I$. Thus, $f(z) > f(c) = d$, so $f(z) \in \{y \in f(I) : y > d\}$. This implies that $z = g(f(z)) \geq L(d)$. It leads to a contradiction because $c < z < L(d)$ by the choice of z . Therefore, $g(d) = L(d)$.

Similarly, we also have a contradiction if $l(d) < g(d)$. Hence $l(d) = g(d)$.

Finally, we want to show that g is continuous at d in the following cases.

If d is an isolated point of $f(I)$, then g is automatically continuous at d .

If d is a right limit point of $f(I)$ but is not a left limit point of $f(I)$, then by Proposition 10.1, we have $g(d) = L(d) = g(d+)$. Therefore, g is continuous at d . Similarly, if d is a left

limit point of $f(I)$ but is not a right limit point of $f(I)$, then we have $g(d) = l(d) = g(d-)$, hence g is continuous at d .

Finally, if d is a right and left limit point of $f(I)$. Then, we have $g(d) = g(d+) = g(d-)$ and so g is continuous at d . The proof is complete. \square

Proposition 10.3 *Let f be an increasing function defined on A and let D be the set of discontinuous points of f . Then D is a countable set.*

Proof: For each integer n , we put $D_n := \{x \in D : n-1 \leq f(x) \leq n\}$. Then $D = \bigcup_{n \in \mathbb{Z}} D_n$. Therefore, it suffices to show that each D_n is countable.

We now fix D_m . By using Proposition 10.1, we first note that $c \in D_m$ if and only if $f(c) - f(c-) > 0$ or $f(c+) - f(c) > 0$. Put $J(c-) := [f(c-), f(c)]$ and $J(c+) := [f(c), f(c+)]$. Then $J(c+)$ or $J(c-)$ is an interval. Therefore, if we put $\alpha(c)$ is the length of $(J(c-) \cup J(c+))$ for $c \in D_m$, then $\alpha(c) > 0$. On the other hand, if $c_1, c_2 \in D_m$ with $c_1 < c_2$, then $J(c_1+) \cap J(c_2-)$ has at most one point if they exist. Thus, we have

$$0 < \sum_{c \in D_m} \alpha(c) \leq m - (m-1) = 1.$$

Since $\alpha(c) > 0$ for all $c \in D_m$, the set D_m need to be countable. In fact, note that we have

$$D_m = \bigcup_{k \in \mathbb{Z}_+} \{c \in D_m : \alpha(c) \geq 1/k\}.$$

Thus, if D_m is uncountable, then there exists a positive integer k so that $R := \{c \in D_m : \alpha(c) \geq 1/k\}$ is infinite. Therefore, $\sum_{c \in R} \alpha(c)$ is infinite. It leads to a contradiction. \square

Let I be an interval. We call a function $f : I \rightarrow \mathbb{R}$ *locally bounded* at $c \in I$ if there is $r > 0$ such that the function f is bounded on $(c-r, c+r) \cap I$. Notice that if $\lim_{x \rightarrow c} f(x)$ exists, then f is locally bounded at c .

Proposition 10.4 *Now if f is a locally bounded at $c \in I$, then the following always exist.*

$$\overline{\lim}_{x \rightarrow c} f(x) := \lim_{\delta \rightarrow 0^+} \left(\sup_{0 < |x-c| < \delta} f(x) \right) \quad \text{and} \quad \underline{\lim}_{x \rightarrow c} f(x) := \lim_{\delta \rightarrow 0^+} \left(\inf_{0 < |x-c| < \delta} f(x) \right).$$

Moreover, we have $\underline{\lim}_{x \rightarrow c} f(x) \leq \overline{\lim}_{x \rightarrow c} f(x)$.

Proof: For simply, we assume that I is an open interval.

For $\delta > 0$, put $L(\delta) := \sup_{0 < |x-c| < \delta} f(x)$ and $\ell(\delta) := \inf_{0 < |x-c| < \delta} f(x)$. Since f is locally bounded at c , $L(\delta)$ and $\ell(\delta)$ are defined for arbitrary small $\delta > 0$. Moreover, the functions $L(\delta)$ is increasing and $\ell(\delta)$ is decreasing. Then by Proposition 10.1, we see that $\overline{\lim}_{x \rightarrow c} f(x) = \inf_{\delta > 0} (\sup_{0 < |x-c| < \delta} f(x))$ and $\underline{\lim}_{x \rightarrow c} f(x) = \sup_{\delta > 0} (\inf_{0 < |x-c| < \delta} f(x))$.

On the other hand, since $\ell(\delta) \leq L(\delta)$ for all $\delta > 0$, we have $\underline{\lim}_{x \rightarrow c} f(x) \leq \overline{\lim}_{x \rightarrow c} f(x)$. \square

Example 10.5 *Define $f(x) = \sin \frac{1}{x}$ for $x \neq 0$. Then $\overline{\lim}_{x \rightarrow 0} f(x) = 1$ and $\underline{\lim}_{x \rightarrow 0} f(x) = -1$. Hence, the case $\underline{\lim}_{x \rightarrow 0} f(x) < \overline{\lim}_{x \rightarrow 0} f(x)$ may occur.*

Proposition 10.6 *Using the notation as in Proposition 10.4, the following are equivalent.*

(i) $\lim_{x \rightarrow c} f(x)$ exists.

(ii) $\underline{\lim}_{x \rightarrow c} f(x) = \overline{\lim}_{x \rightarrow c} f(x)$.

In this case, we have $\lim_{x \rightarrow c} f(x) = \underline{\lim}_{x \rightarrow c} f(x) = \overline{\lim}_{x \rightarrow c} f(x)$.

Proof: Let $L := \lim_{x \rightarrow c} f(x)$ if it exists.

For showing (i) \Rightarrow (ii), let $\varepsilon > 0$, then there is $\delta > 0$ such that $L - \varepsilon < f(x) < L + \varepsilon$ whenever $x \in I$ with $0 < |x - c| < \delta$. Using the notation as in Proposition 10.4, then we have $L - \varepsilon \leq \ell(\delta_1) \leq L(\delta_1) \leq L + \varepsilon$ for all $0 < \delta_1 < \delta$. Taking $\delta_1 \rightarrow 0+$, we have

$$L - \varepsilon \leq \underline{\lim}_{x \rightarrow c} f(x) \leq \overline{\lim}_{x \rightarrow c} f(x) \leq L + \varepsilon$$

for all $\varepsilon > 0$. This gives $\underline{\lim}_{x \rightarrow c} f(x) = \overline{\lim}_{x \rightarrow c} f(x) = L$ as desired.

(ii) \Rightarrow (i) follows immediately from the simple observation that $\ell(\delta) \leq f(x) \leq L(\delta)$ for all $\delta > 0$ and for all $x \in I$ with $0 < |x - c| < \delta$. \square

Definition 10.7 *A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be of bounded variation if it satisfies the following condition.*

$$\|f\|_{BV} := \sup \left\{ \sum_{k=1}^n |f(x_k) - f(x_{k-1})| : a = x_0 < \dots < x_n = b \right\} < \infty.$$

We call $\|f\|_{BV}$ the total variation of f and write $BV[a, b]$ for the set of all functions of bounded variation.

Proposition 10.8 *We keep all notation as given above. Then we have*

(i) *The space $BV[a, b]$ is a vector space.*

(ii) *Every function of bounded variation is bounded.*

(iii) *Every monotone function is of bounded variation.*

(iv) *Every Lipschitz function is of bounded variation.*

Proof: Parts (i), (iii) and (iv) are clear.

For showing Part (ii), let $f \in BV[a, b]$ and $x \in (a, b)$. Then by considering the partition $a < x < b$, we have $|f(x) - f(a)| + |f(b) - f(x)| \leq \|f\|_{BV}$. This gives $2|f(x)| \leq |f(a)| + |f(b)| + \|f\|_{BV}$ for all $x \in (a, b)$ and hence, f is bounded on $[a, b]$. \square

Remark 10.9 The part (iii) in Proposition 10.8 tells us that a function of bounded variation may not be continuous. On the other hand, a bounded function is not necessary to be of bounded variation.

Example 10.10 *Let*

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \in (0, \frac{2}{\pi}]; \\ 0 & \text{if } x = 0. \end{cases}$$

Notice that f is bounded on $[0, \frac{2}{\pi}]$ but $f \notin BV[0, \frac{2}{\pi}]$. In fact, for each positive integer n , if we let $x_k := \frac{1}{\frac{\pi}{2} + (n-k)\pi}$ for $1 \leq k \leq n$ and consider the partition on $[0, \frac{2}{\pi}] : 0 = x_0 < x_1 < \dots < x_{n-1} < x_n = \frac{2}{\pi}$, then we have

$$\sum_{k=1}^n |f(x_k) - f(x_{k-1})| \geq \sum_{k=2}^n |f(x_k) - f(x_{k-1})| \geq 2(n-1)$$

for all $n \geq 2$. This implies that f is not of bounded variation.

Proposition 10.11 *Suppose that $f \in BV[a, b]$. Put*

$$\phi(x) = \begin{cases} \sup\{\sum_{k=1}^n |f(x_k) - f(x_{k-1})| : a = x_0 < \dots < x_n = x\} & \text{if } x \in (a, b]; \\ 0 & \text{if } x = a. \end{cases}$$

Then the functions ϕ and $\phi - f$ are increasing on $[a, b]$. Consequently, a function $f : [a, b] \rightarrow \mathbb{R}$ is of bounded variation if and only if $f = u - v$ for some increasing functions u and v on $[a, b]$.

Proof: We first notice that since $f \in BV[a, b]$, the function ϕ is well defined. In fact, we have $0 \leq \phi(x) \leq \|f\|_{BV}$ for all $x \in [a, b]$. Let $a < x < y \leq b$. For any partition on $[a, x] : a = x_0 < \dots < x_n = x$, we see that

$$\sum_{k=1}^n |f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^n |f(x_k) - f(x_{k-1})| + |f(y) - f(x)| \leq \phi(y).$$

This implies that $\phi(x) \leq \phi(y)$ and

$$\phi(x) \leq \phi(y) - |f(y) - f(x)| \leq \phi(y) - (f(y) - f(x)).$$

Thus, we have

$$\phi(x) - f(x) \leq \phi(y) - f(y).$$

Therefore, the function $\phi - f$ is increasing on $[a, b]$.

For the last assertion, if $f \in BV[a, b]$, then we have $f = \phi - (\phi - f)$ and so f can be written as the difference of increasing functions as desired. Conversely, since $BV[a, b]$ is a vector space and every increasing function is of bounded variation, the difference of increasing functions is a function of bounded variation. The proof is complete. \square

References

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